PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 1, May 1994

CENTRAL EXTENSIONS OF NONSYMMETRIZABLE KAC-MOODY ALGEBRAS OVER COMMUTATIVE ALGEBRAS

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(Communicated by Roe W. Goodman)

ABSTRACT. For a commutative algebra R over a field k of characteristic zero and a nonsymmetrizable Kac-Moody algebra $g(A)$, we prove that the Lie algebra $g_R(A) = R \otimes_k g(A)$ is centrally closed. Consequently, we get a characterization of the symmetrizability of $g(A)$ by the second homology group of the Kac-Moody algebra over Laurent polynomials. Also a presentation of $g_R(A)$ is given when A is of nonaffine type.

1. INTRODUCTION

Let k be a field of characteristic zero and R be an associative and commutative k-algebra with identity. Given an $l \times l$ generalized Cartan matrix (GCM, for short) $A = (A_{ij})$, the associated Lie algebra $g(A)$ over k is called a Kac-Moody algebra and has the following standard presentation:

generators: e_i, f_i, h_i, $1 \le i \le l$;

relations: $[h_i, e_j] = A_{ij}e_j$, $[h_i, f_j] = -A_{ij}f_j$, $[h_i, h_j] = 0$, $[e_i, f_i] = h_i$, and

for $i \neq j$, $[e_i, f_j] = (ade_i)^{-A_{ij}+1}e_j = (adf_i)^{-A_{ij}+1}f_j = 0$.

Consider the k-vector space $g_R(A) = R \otimes_k g(A)$ and define a Lie bracket on $g_R(A)$ by

$$[a \otimes x, b \otimes y] = ab \otimes [x, y],$$

where $a, b \in R$, $x, y \in g(A)$. Then $g_R(A)$ is a perfect Lie algebra over k.

Kassel [7] studied the universal central extensions of $g_R(A)$ when A is of finite type. An elegant approach to this work, which follows Wilson [10], has been given by Moody, Rao, and Yokonuma in [9]. Some of these results are used in [3]. More generally, if A is indecomposable and symmetrizable, Haddi [5] computed the second homology groups of $g_R(A)$ as follows:

$H_2(g_R(A)) = \Omega_R^1/dR$, if A is of nonaffine type,

$H_2(g_R(A)) = \Omega_R^1/dR \oplus (\Omega_R^1 \otimes I)$, if A is of affine type,

where Ω_R^1/dR is the module of Kähler differentials of R over k modulo exact forms and I is the augmentation ideal of the algebra $k[t, t^{-1}]$ of Laurent polynomials which is generated by $(t-1)$.

Received by the editors April 30, 1992 and, in revised form, August 31, 1992.
1991 *Mathematics Subject Classification*. Primary 17B67; Secondary 17B55, 17B60.

In this note, we will give a presentation of $g_R(A)$ for a nonaffine GCM A and show that $H_2(g_R(A)) = 0$ when A is indecomposable and nonsymmetrizable. In particular, it follows that if A is an indecomposable GCM, and $R = k[t, t^{-1}]$, then $H_2(g_R(A)) \neq 0$ if and only if A is symmetrizable. This means that the second homology group of a Kac-Moody algebra over Laurent polynomials characterizes the symmetrizability of the Kac-Moody algebra.

2. Presentations for $g_R(A)$

All Lie algebras under consideration will be regarded as k-Lie algebras. We write $x(a) = a \otimes x$ for $a \in R$, $x \in g(A)$, so then $g_R(A)$ is a Lie algebra generated by $e_i(a)$, $f_i(a)$, $h_i(a)$ for all $a \in R$, $1 \leq i \leq l$. As in [4, 6, or 8], for each tuple $(n_1, \dots, n_l)$ of nonnegative (resp. nonpositive) integers not all zero, let $g(n_1, \dots, n_l)$ be the subspace of $g(A)$ spanned by the elements

$$[e_{i_1}, \dots [e_{i_{m-1}}, e_{i_m}] \cdots] \quad (\text{resp. } [f_{i_1}, \dots [f_{i_{m-1}}, f_{i_m}] \cdots]),$$

where e_j (resp. f_j) occurs $|n_j|$ times. Also let $g(0, \dots, 0)$ be the linear span of $h_1, \dots, h_l$ and $g(n_1, \dots, n_l) = 0$ for any other tuple of integers. Then

$$g(A) = \coprod_{(n_1, \dots, n_l) \in Z^l} g(n_1, \dots, n_l),$$

which is a Lie algebra gradation of $g(A)$. Obviously, this induces a gradation of $g_R(A)$,

$$g_R(A) = \coprod_{(n_1, \dots, n_l) \in Z^l} g_R(n_1, \dots, n_l),$$

where $g_R(n_1, \dots, n_l) = R \otimes g(n_1, \dots, n_l) = \{x(a) \colon x \in g(n_1, \dots, n_l),\ a \in R\}$.

For convenience, we write $[x_1, [x_2, \dots, [x_{m-1}, x_m] \cdots]] = [x_1, x_2, \dots, x_{m-1}, x_m]$.

Let $\overline{g}_R(A)$ be the Lie algebra over k with the following presentation:
generators: $E_i(a)$, $F_i(a)$, $H_i(a)$, $a \in R$, $1 \leq i \leq l$;
relations:

$$E_i(\alpha a + \beta b) = \alpha E_i(a) + \beta E_i(b), \qquad F_i(\alpha a + \beta b) = \alpha F_i(a) + \beta F_i(b), \tag{1}$$

$$[H_i(a), E_j(b)] = A_{ij} E_j(ab), \tag{2}$$

$$[H_i(a), F_j(b)] = -A_{ij} F_j(ab), \tag{3}$$

$$[E_i(a), F_i(b)] = H_i(ab), \tag{4}$$

$$[E_i(a), E_i(b)] = 0, \tag{5}$$

$$[F_i(a), F_i(b)] = 0, \tag{6}$$

$$\begin{aligned} &[E_{i_1}(a_1), \dots [E_{i_{m-1}}(a_{m-1}), E_{i_m}(a_m)] \cdots] \\ &\quad = [E_{i_1}(1), \dots [E_{i_{m-1}}(1), E_{i_m}(a_1 \cdots a_{m-1} a_m)] \cdots], \end{aligned} \tag{7}$$

$$\begin{aligned} &[F_{i_1}(a_1), \dots [F_{i_{m-1}}(a_{m-1}), F_{i_m}(a_m)] \cdots] \\ &\quad = [F_{i_1}(1), \dots [F_{i_{m-1}}(1), F_{i_m}(a_1 \cdots a_{m-1} a_m)] \cdots], \end{aligned} \tag{8}$$

for all $a_1, \dots, a_m$, $a, b \in R$, $\alpha, \beta \in k$, $1 \le i_1, \dots, i_m$, $i, j \le l$, and if $i \ne j$,

$$[E_i(a), F_j(b)] = 0, \tag{9}$$

$$(adE_i(1))^{-A_{ij}+1} E_j(a) = 0, \tag{10}$$

$$(adF_i(1))^{-A_{ij}+1} F_j(a) = 0, \tag{11}$$

where $a, b \in R$, $1 \le i \ne j \le l$.

From the definition, it is not hard to check that $H_i(\alpha a + \beta b) = \alpha H_i(a) + \beta H_i(b)$, and

$$[H_i(a), H_j(b)] = 0, \tag{12}$$

$$\begin{aligned} &[H_j(a), [E_{i_1}(1), \dots, E_{i_{m-1}}(1), E_{i_m}(b)]] \\ &\quad = (A_{ji_1} + \cdots + A_{ji_m})[E_{i_1}(1), \dots, E_{i_{m-1}}(1), E_{i_m}(ab)], \end{aligned} \tag{13}$$

$$\begin{aligned} &[H_j(a), [F_{i_1}(1), \dots, F_{i_{m-1}}(1), F_{i_m}(b)]] \\ &\quad = -(A_{ji_1} + \cdots + A_{ji_m})[F_{i_1}(1), \dots, F_{i_{m-1}}(1), F_{i_m}(ab)], \end{aligned} \tag{14}$$

where $a, b \in R$, $\alpha, \beta \in k$, $1 \le i_1, \dots, i_m$, $i, j \le l$.

One can easily see that the Lie algebra $g_R(A)$ satisfies the above defining relations of $\overline{g}_R(A)$ (replacing $E_i(a)$, $F_i(a)$, $H_i(a)$ by $e_i(a)$, $f_i(a)$, $h_i(A)$, respectively). By the universal property of $\overline{g}_R(A)$, there exists a homomorphism φ from $\overline{g}_R(A)$ onto $g_R(A)$ such that

$$\varphi(E_i(a)) = e_i(a), \qquad \varphi(F_i(a)) = f_i(a), \quad \varphi(H_i(a)) = h_i(a)$$

for all $a \in R$, $1 \le i \le l$.

Let $\overline{g}(A)$ be the subalgebra of $\overline{g}_R(A)$ generated by $\{E_i(1), F_i(1), H_i(1), 1 \le i \le l\}$. The restriction of φ on $\overline{g}(A)$ induces a homomorphism from $\overline{g}(A)$ onto $g(A)$ which we shall denote by ψ. Since $g(A)$ has a universal property, there exists a homomorphism ρ from $g(A)$ onto $\overline{g}(A)$ such that

$$\rho(e_i) = E_i(1), \quad \rho(f_i) = F_i(1), \quad \rho(h_i) = H_i(1), \qquad \text{for } 1 \le i \le l.$$

It follows that $\rho \circ \psi$ is the identity map, proving that ψ is an isomorphism.

Fix a basis $\{r_\lambda\}_{\lambda \in \Lambda}$ of R with $1 \in \{r_\lambda\}_{\lambda \in \Lambda}$, where Λ is an index set. Then

$$g_R(A) = \coprod_{(n_1, \dots, n_l) \in Z^l} \left(\coprod_{\lambda \in \Lambda} r_\lambda \otimes g(n_1, \dots, n_l) \right). \tag{15}$$

Now we assume that A is of nonaffine type and recall the following well-known fact (see Theorem 4.3 in [6] or Proposition 3.6.5 in [8]).

Lemma 1. *If $A = (A_{ij})$ is an indecomposable and nonaffine GCM, and $n_1, \dots, n_l$ are nonnegative integers such that*

$$\begin{aligned} A_{11} n_1 + \cdots + A_{1l} n_l &= 0 \\ &\vdots \\ A_{1l} n_1 + \cdots + A_{ll} n_l &= 0, \end{aligned}$$

then $n_1 = \cdots = n_l = 0$.

To define a linear mapping from $g_R(A)$ to $\overline{g}_R(A)$, we need the following elementary result.

Lemma 2. *Suppose V is any vector space over k with $\{x_i\}_{i\in I}$ spanning V and let W be another vector space over k with elements $\{y_i\}_{i\in I} \subseteq W$. If $\sum_i \alpha_i x_i = 0$ implies $\sum_i \alpha_i y_i = 0$, then there is a unique linear map ϕ from V to W satisfying $\phi(\sum_i \alpha_i x_i) = \sum_i \alpha_i y_i$.*

We define a mapping $\phi\colon g_R(A) \to \overline{g}_R(A)$ given by

$$\phi\left(\sum_{i=1}^{l} \alpha_i h_i(r_\lambda)\right) = \sum_{i=1}^{l} \alpha_i H_i(r_\lambda),$$

$$\phi\left(\sum_{\sigma\in S_m} \alpha_\sigma[e_{i_{\sigma(1)}}(1), \dots, e_{i_{\sigma(m-1)}}(1), e_{i_{\sigma(m)}}(r_\lambda)]\right)$$
$$= \sum_{\sigma\in S_m} \alpha_\sigma[E_{i_{\sigma(1)}}(1), \dots, E_{i_{\sigma(m-1)}}(1), E_{i_{\sigma(m)}}(r_\lambda)],$$

$$\phi\left(\sum_{\sigma\in S_m} \alpha_\sigma[f_{i_{\sigma(1)}}(1), \dots, f_{i_{\sigma(m-1)}}(1), f_{i_{\sigma(m)}}(r_\lambda)]\right)$$
$$= \sum_{\sigma\in S_m} \alpha_\sigma[F_{i_{\sigma(1)}}(1), \dots, F_{i_{\sigma(m-1)}}(1), F_{i_{\sigma(m)}}(r_\lambda)],$$

where $\alpha_\sigma \in k$, and S_m is the standard permutation group on $\{1, \dots, m\}$.

We claim that ϕ is a well-defined k-linear mapping. Note that by (15) and Lemma 2, it suffices to show that if

$$\sum_{\sigma\in S_m} \alpha_\sigma[e_{i_{\sigma(1)}}(1), \dots, e_{i_{\sigma(m-1)}}(1), e_{i_{\sigma(m)}}(r_\lambda)] = 0, \tag{16}$$

where $[e_{i_1}(1), \dots, e_{i_{m-1}}(1), e_{i_m}(r_\lambda)] \in r_\lambda \otimes g(n_1, \dots, n_l)$ and $n_1, \dots, n_l$ are nonnegative integers not all zero, then

$$\sum_{\sigma\in S_m} \alpha_\sigma[E_{i_{\sigma(1)}}(1), \dots, E_{i_{\sigma(m-1)}}(1), E_{i_{\sigma(m)}}(r_\lambda)] = 0. \tag{17}$$

Indeed, (16) gives us

$$\sum_{\sigma\in S_m} \alpha_\sigma[e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(m-1)}}, e_{i_{\sigma(m)}}] = 0.$$

Since ψ is an isomorphism from $\overline{g}(A)$ onto $g(A)$, we have

$$\sum_{\sigma\in S_m} \alpha_\sigma[E_{i_{\sigma(1)}}(1), \dots, E_{i_{\sigma(m-1)}}(1), E_{i_{\sigma(m)}}(1)] = 0.$$

Let $C = \sum_{\sigma\in S_m} \alpha_\sigma[E_{i_{\sigma(1)}}(1), \dots, E_{i_{\sigma(m-1)}}(1), E_{i_{\sigma(m)}}(r_\lambda)]$. Then for any j, by (13), we get

$$\begin{aligned}
[H_j(1), C] &= \left[H_j(1), \sum_{\sigma\in S_m} \alpha_\sigma[E_{i_{\sigma(1)}}(1), \dots, E_{i_{\sigma(m-1)}}(1), E_{i_{\sigma(m)}}(r_\lambda)]\right] \\
&= \sum_{\sigma\in S_m} \alpha_\sigma[H_j(1), [E_{i_{\sigma(1)}}(1), \dots, E_{i_{\sigma(m-1)}}(1), E_{i_{\sigma(m)}}(r_\lambda)]] \\
&= \sum_{\sigma\in S_m} \alpha_\sigma(A_{j1}n_1 + \dots + A_{jl}n_l)[E_{i_{\sigma(1)}}(1), \dots, E_{i_{\sigma(m-1)}}(1), E_{i_{\sigma(m)}}(r_\lambda)]] \\
&= (A_{j1}n_1 + \dots + A_{jl}n_l)C.
\end{aligned}$$

Also,

$$\begin{aligned}[H_j(1), C] &= \left[H_j(1), \sum_{\sigma\in S_m} \alpha_\sigma[E_{i_{\sigma(1)}}(1), \dots, E_{i_{\sigma(m-1)}}(1), E_{i_{\sigma(m)}}(r_\lambda)]\right] \\ &= \sum_{\sigma\in S_m} \alpha_\sigma[H_j(r_\lambda), [E_{i_{\sigma(1)}}(1), \dots, E_{i_{\sigma(m-1)}}(1), E_{i_{\sigma(m)}}(1)]] \\ &= \left[H_j(r_\lambda), \sum_{\sigma\in S_m} \alpha_\sigma[E_{i_{\sigma(1)}}(1), \dots, E_{i_{\sigma(m-1)}}(1), E_{i_{\sigma(m)}}(1)]\right] = 0.\end{aligned}$$

Thus $C = 0$ which verifies (17), for otherwise $A_{j1}n_1 + \cdots + A_{jl}n_l = 0$ for all $1 \le j \le l$ and this contradicts Lemma 1.

Lemma 3. *Let L_1 and L_2 be two Lie algebras over k. Suppose L_1 is generated by $\{x_i\}_{i\in I}$. If $\phi\colon L_1 \to L_2$ is a k-linear mapping and $\phi([x_i, y]) = [\phi(x_i), \phi(y)]$ for all $i \in I$, $y \in L_1$, then ϕ is a Lie algebra homomorphism.*

Proof. This is a routine matter.

Clearly, $g_R(A)$ is actually generated by $e_i(a)$, $f_i(a)$, $a \in R$, $1 \le i \le l$. We will show

$$\phi([e_j(a), y]) = [\phi(e_j(a)), \phi(y)], \qquad \phi([f_j(a), y]) = [\phi(f_j(a)), \phi(y)]$$

for all $a \in R$, $y \in g_R(A)$, $1 \le j \le l$.

Case 1. Let $y \in g_R(0, \dots, 0)$. Using (2), we obtain

$$\begin{aligned}\phi([e_j(a), h_i(b)]) &= \phi(-A_{ij}e_j(ab)) = -A_{ij}\phi(e_j(ab)) \\ &= -A_{ij}E_j(ab) = [E_j(a), H_i(b)] = [\phi(e_j(a)), \phi(h_i(b))].\end{aligned}$$

Hence $\phi([e_j(a), y]) = [\phi(e_j(a)), \phi(y)]$ for all $y \in g_R(0, \dots, 0)$.

Case 2. Similar to Case 1, $\phi([f_j(a), y]) = [\phi(f_j(a)), \phi(y)]$ for all $y \in g_R(0, \dots, 0)$.

Case 3. Using (7), we get

$$\begin{aligned}\phi([e_j(a), e_i(b)]) &= \phi([e_j(1), e_i(ab)]) = [E_j(1), E_i(ab)] \\ &= [E_j(a), E_i(b)] = [\phi(e_j(a)), \phi(e_i(b))].\end{aligned}$$

By induction on m, one can see

$$\begin{aligned}&\phi([e_j(a), [e_{i_1}(1), \dots, e_{i_{m-1}}(1), e_{i_m}(b)]]) \\ &\quad = [\phi(e_j(a)), \phi([e_{i_1}(1), \dots, e_{i_{m-1}}(1), e_{i_m}(b)])].\end{aligned}$$

Hence $\phi([e_j(a), y]) = [\phi(e_j(a)), \phi(y)]$ for all $y \in g_R(n_1, \dots, n_l)$, where $n_1, \dots, n_l$ are nonnegative integers not all zero.

Case 4. Similar to Case 3, $\phi([f_j(a), y]) = [\phi(f_j(a)), \phi(y)]$ for all $y \in g_R(n_1, \dots, n_l)$, where $n_1, \dots, n_l$ are nonpositive integers not all zero.

Case 5. Let $y \in g_R(n_1, \dots, n_l)$, where $n_1, \dots, n_l$ are nonpositive integers not all zero. When $n_1 + \cdots + n_l = n = 1$, using (4) and (9), we obtain

$$\phi([e_j(a), f_i(b)]) = [\phi(e_j(a)), \phi(f_i(b))].$$

Next, if $n_1 + \cdots + n_l = n \ge 2$, we have

$$\begin{aligned}&\phi([e_j(a), [f_{i_1}(1), \dots, f_{i_{m-1}}(1), f_{i_m}(b)]]) \\ &\quad = \phi([[e_j(a), f_{i_1}(1)], [f_{i_2}(1), \dots, f_{i_{m-1}}(1), f_{i_m}(b)]]) \\ &\qquad + \phi([f_{i_1}(1), [e_j(a), [f_{i_2}(1), \dots, f_{i_{m-1}}(1), f_{i_m}(b)]]]).\end{aligned}$$

Now, using (14),

$$\begin{aligned}
&\phi([[e_j(a), f_{i_1}(1)], [f_{i_2}(1), \dots, f_{i_{m-1}}(1), f_{i_m}(b)]]) \\
&\quad = \phi([\delta_{ji_1} h_j(a), [f_{i_2}(1), \dots, f_{i_{m-1}}(1), f_{i_m}(b)]]) \\
&\quad = -\delta_{ji_1}(A_{ji_2} + \cdots + A_{ji_m})\phi([f_{i_2}(1), \dots, f_{i_{m-1}}(1), f_{i_m}(ab)]) \\
&\quad = -\delta_{ji_1}(A_{ji_2} + \cdots + A_{ji_m})[F_{i_2}(1), \dots, F_{i_{m-1}}(1), F_{i_m}(ab)]) \\
&\quad = \delta_{ji_1}[H_j(a), [F_{i_2}(1), \dots, F_{i_{m-1}}(1), F_{i_m}(b)]] \\
&\quad = [[E_j(a), F_{i_1}(1)], [F_{i_2}(1), \dots, F_{i_{m-1}}(1), F_{i_m}(b)]]
\end{aligned}$$

By Case 2, Case 4, and induction on n, we get

$$\begin{aligned}
&\phi([f_{i_1}(1), [e_j(a), [f_{i_2}(1), \dots, f_{i_{m-1}}(1), f_{i_m}(b)]]]) \\
&\quad = [\phi(f_{i_1}(1)), \phi([e_j(a), [f_{i_2}(1), \dots, f_{i_{m-1}}(1), f_{i_m}(b)]])] \\
&\quad = [F_{i_1}(1), [E_j(a), [F_{i_2}(1), \dots, F_{i_{m-1}}(1), F_{i_m}(b)]]].
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&\phi([e_j(a), [f_{i_1}(1), \dots, f_{i_{m-1}}(1), f_{i_m}(b)]]) \\
&\quad = [[E_j(a), F_{i_1}(1)], [F_{i_2}(1), \dots, F_{i_{m-1}}(1), F_{i_m}(b)]] \\
&\qquad + [F_{i_1}(1), [E_j(a), [F_{i_2}(1), \dots, F_{i_{m-1}}(1), F_{i_m}(b)]]] \\
&\quad = [E_j(a), [F_{i_1}(1), \dots, F_{i_{m-1}}(1), F_{i_m}(b)]] \\
&\quad = [\phi(e_j(a)), \phi([f_{i_1}(1), \dots, f_{i_{m-1}}(1), f_{i_m}(b)])].
\end{aligned}$$

Therefore $\phi([e_j(a), y]) = [\phi(e_j(a)), \phi(y)]$ for all $y \in g_R(n_1, \dots, n_l)$ where $n_1, \dots, n_l$ are nonpositive integers not all zero.

Case 6. Similar to Case 5, $\phi([f_j(a), y]) = [\phi(f_j(a)), \phi(y)]$ for all $y \in g_R(n_1, \dots, n_l)$, where $n_1, \dots, n_l$ are nonnegative integers not all zero.

By Lemma 3, ϕ is a homomorphism from $g_R(A)$ onto $\overline{g}_R(A)$. Clearly $\phi \circ \varphi = id_{\overline{g}_R(A)}$, so φ is injective.

Summarizing the above, we have proved

Theorem 1. *If A is an indecomposable and nonaffine GCM, then $g_R(A)$ is isomorphic to $\overline{g}_R(A)$.*

3. Central extensions of $g_R(A)$

Some ideas of this section are motivated by the work of Benkart and Moody [1].

Theorem 2. *If A is an indecomposable and nonsymmetrizable GCM, then each central extension of the Lie algebra $g_R(A)$ splits.*

Proof. Suppose that $0 \to V \to L \xrightarrow{\pi} g_R(A) \to 0$ is a central extension of $g_R(A)$.

Let $\tilde{e}_i(a)$, $\tilde{h}_i(a)$, and $\tilde{f}_i(a)$, for $1 \le i \le l$, be preimages of $e_i(a)$, $h_i(a)$, and $f_i(a)$, respectively, in L under π. It is sufficient to choose $\tilde{e}_i(a)$, $\tilde{h}_i(a)$, $\tilde{f}_i(a)$ for $1 \le i \le l$, $a \in \{r_\lambda\}_{\lambda \in \Lambda}$, and then extend our choices to all $a \in R$ by linearity.

Clearly, $[\tilde{h}_i(1), \tilde{e}_i(a)] = 2\tilde{e}_i(a) + c_i(a)$ for some $c_i(a) \in V$. Replacing $\tilde{e}_i(a)$ by $\tilde{e}_i(a) + \frac{1}{2}c_i(a)$, we get

$$[\tilde{h}_i(1), \tilde{e}_i(a)] = 2\tilde{e}_i(a) \quad \text{for } 1 \le i \le l, \ a \in R. \tag{18}$$

Similarly we may take $\tilde{f}_i(a)$, $\tilde{h}_i(a)$ so that

$$[\tilde{h}_i(1), \tilde{f}_i(a)] = -2\tilde{f}_i(a), \tag{19}$$

$$[\tilde{e}_i(1), \tilde{f}_i(a)] = \tilde{h}_i(a) \tag{20}$$

for $1 \leq i \leq l$, $a \in R$.

The proof of the theorem will be carried out in steps.

Step 1. We claim

$$[\tilde{h}_i(a), \tilde{e}_j(b)] = A_{ij}\tilde{e}_j(ab), \tag{21}$$

$$[\tilde{h}_i(a), \tilde{f}_j(b)] = -A_{ij}\tilde{f}_j(ab), \tag{22}$$

$$[\tilde{e}_i(a), \tilde{e}_i(b)] = 0, \tag{23}$$

$$[\tilde{f}_i(a), \tilde{f}_i(b)] = 0, \tag{24}$$

for all $a, b \in R$, $1 \leq i, j \leq l$, and if $i \neq j$,

$$[\tilde{e}_i(a), \tilde{f}_j(b)] = 0, \tag{25}$$

$$(ad\tilde{e}_i(1))^{-A_{ij}+1}\tilde{e}_j(a) = 0, \tag{26}$$

$$(ad\tilde{f}_i(1))^{-A_{ij}+1}\tilde{f}_j(a) = 0, \tag{27}$$

where $a, b \in R$, $1 \leq i \neq j \leq l$.

Using (18) and the Jacobi identity, we have

$$\begin{aligned}[\tilde{h}_i(a), \tilde{e}_j(b)] &= [\tilde{h}_i(a), \tfrac{1}{2}[\tilde{h}_j(1), \tilde{e}_j(b)]] \\ &= \tfrac{1}{2}[[\tilde{h}_i(a), \tilde{h}_j(1)], \tilde{e}_j(b)] + \tfrac{1}{2}[\tilde{h}_i(1), [\tilde{h}_i(a), \tilde{e}_j(b)]].\end{aligned}$$

Applying the central trick, namely, $[x, y] = [x', y']$ if $\pi(x) = \pi(x')$, $\pi(y) = \pi(y')$ for $x, y, x', y' \in L$, we obtain $[[\tilde{h}_i(a), \tilde{h}_j(1)], \tilde{e}_j(b)] = 0$ and $[\tilde{h}_j(1), [\tilde{h}_i(a), \tilde{e}_j(b)]] = [\tilde{h}_j(1), A_{ij}\tilde{e}_j(ab)] = 2A_{ij}\tilde{e}_j(ab)$.

Thus $[\tilde{h}_i(a), \tilde{e}_j(b)] = A_{ij}\tilde{e}_j(ab)$ which gives (21). Similarly (22) holds.

Let $i \neq j$. Since $[\tilde{e}_i(a), \tilde{f}_j(b)] = c \in V$, we see that

$$\begin{aligned}0 &= [\tilde{h}_i(1), [\tilde{e}_i(a), \tilde{f}_j(b)]] = [[\tilde{h}_i(1), \tilde{e}_i(a)], \tilde{f}_j(b)] + [\tilde{e}_i(a), [\tilde{h}_i(1), \tilde{f}_j(b)]] \\ &= [2\tilde{e}_i(a), \tilde{f}_j(b)] - [\tilde{e}_i(a), A_{ij}\tilde{f}_j(b)] = (2 - A_{ij})c\end{aligned}$$

which implies $c = 0$ as $A_{ij} \leq 0$, so (25) follows.

To prove (23), (24), (26), and (27), we need the following formulas which follow easily from our definition and induction.

$$\begin{aligned}&[\tilde{h}_j(1), [\tilde{e}_{i_1}(a_1), \dots, \tilde{e}_{i_m}(a_m)]] \\ &\quad = (A_{ji_1} + \cdots + A_{ji_m})[\tilde{e}_{i_1}(a_1), \dots, \tilde{e}_{i_m}(a_m)]\end{aligned} \tag{28}$$

$$\begin{aligned}&[\tilde{h}_j(1), [\tilde{f}_{i_1}(a_1), \dots, \tilde{f}_{i_m}(a_m)]] \\ &\quad = -(A_{ji_1} + \cdots + A_{ji_m})[\tilde{f}_{i_1}(a_1), \dots, \tilde{f}_{i_m}(a_m)],\end{aligned} \tag{29}$$

where $1 \leq j$, $i_1, \dots, i_m \leq l$ and $a_1, \dots, a_m \in R$.

Now it is easy to see that $[\tilde{e}_i(a), \tilde{e}_i(b)] \in V$, so

$$0 = [\tilde{h}_i(1), [\tilde{e}_i(a), \tilde{e}_i(b)]] = 4[\tilde{e}_i(a), \tilde{e}_i(b)]$$

which gives (23). Similarly we have (24).

Also, if $i \neq j$, $(ad\tilde{e}_i(1))^{-A_{ij}+1}\tilde{e}_j(a) = c \in V$, then

$$0 = [\tilde{h}_i(1), c] = (2(-A_{ij}+1) + A_{ij})c = (-A_{ij}+2)c$$

which implies $c = 0$ as $A_{ij} \leq 0$. Thus (26) holds and so does (27).

Step 2. We claim

$$\begin{aligned}(30)\qquad &[\tilde{e}_{i_1}(a_1), \dots, \tilde{e}_{i_{m-1}}(a_{m-1}), \tilde{e}_{i_m}(a_m)] \\ &\quad = [\tilde{e}_{i_1}(1), \dots, \tilde{e}_{i_{m-1}}(1), \tilde{e}_{i_m}(a_1 \cdots a_m)],\end{aligned}$$

$$\begin{aligned}(31)\qquad &[\tilde{f}_{i_1}(a_1), \dots, \tilde{f}_{i_{m-1}}(a_{m-1}), \tilde{f}_{i_m}(a_m)] \\ &\quad = [\tilde{f}_{i_1}(1), \dots, \tilde{f}_{i_{m-1}}(1), \tilde{f}_{i_m}(a_1 \cdots a_m)]\end{aligned}$$

for all $a_1, \dots, a_m \in R$, $1 \leq i_1, \dots, i_m \leq l$, and $m \geq 1$.

In fact, suppose j occurs n_j times among $\{i_1, \dots, i_m\}$. Obviously, we have

$$[\tilde{e}_{i_1}(a_1), \dots, \tilde{e}_{i_{m-1}}(a_{m-1}), \tilde{e}_{i_m}(a_m)] - [\tilde{e}_{i_1}(1), \dots, \tilde{e}_{i_{m-1}}(1), \tilde{e}_{i_m}(a_1 \cdots a_m)] = c$$

for some $c \in V$. Letting $\tilde{h}_j(1)$ act on both sides, by (28), we get

$$(A_{j1}n_1 + \cdots + A_{jl}n_l)c = 0 \quad \text{for all } j.$$

$A = (A_{ij})$ is nonsymmetrizable; it is, of course, of nonaffine type, so Lemma 1 forces $c = 0$. Thus (30) holds and so does (31).

Step 3. We claim

$$(32)\qquad [\tilde{h}_i(a), \tilde{h}_j(b)] = 0,$$

$$(33)\qquad [\tilde{e}_i(a), \tilde{f}_i(b)] = \tilde{h}_i(ab)$$

for all $a, b \in R$, $1 \leq i, j \leq l$.

Let $[\tilde{h}_i(a), \tilde{h}_j(b)] = H_{ij}(a, b)$, evidently $H_{ij}(a, b) \in V$, and

$$(34)\qquad H_{ij}(a, b) + H_{ji}(b, a) = 0.$$

By (20), (21), and the Jacobi identity, one has

$$\begin{aligned}(35)\qquad [\tilde{e}_i(a), \tilde{f}_i(b)] &= [\tfrac{1}{2}[\tilde{h}_i(a), \tilde{e}_i(1)], \tilde{f}_i(b)] \\ &= \tfrac{1}{2}[\tilde{h}_i(a), [\tilde{e}_i(1), \tilde{f}_i(b)]] - \tfrac{1}{2}[\tilde{e}_i(1), [\tilde{h}_i(a), \tilde{f}_i(b)]] \\ &= \tfrac{1}{2}[\tilde{h}_i(a), \tilde{h}_j(b)] + [\tilde{e}_i(1), \tilde{f}_i(ab)] = \tilde{h}_i(ab) + \tfrac{1}{2}H_{ii}(a, b).\end{aligned}$$

From (35), we find

$$\begin{aligned}(36)\qquad H_{ij}(c, ab) &= [\tilde{h}_i(c), \tilde{h}_j(ab)] = [\tilde{h}_i(c), [\tilde{e}_j(a), \tilde{f}_j(b)]] \\ &= [[\tilde{h}_i(c), \tilde{e}_j(a)], \tilde{f}_j(b)] + [\tilde{e}_j(a), [\tilde{h}_i(c), \tilde{f}_j(b)]] \\ &= A_{ij}[\tilde{e}_j(ca), \tilde{f}_j(b)] - A_{ij}[\tilde{e}_j(a), \tilde{f}_j(cb)] \\ &= A_{ij}(\tilde{h}_j(abc) + \tfrac{1}{2}H_{jj}(ca, b)) - A_{ij}(\tilde{h}_j(abc) + \tfrac{1}{2}H_{jj}(a, cb)) \\ &= \tfrac{1}{2}A_{ij}H_{jj}(ca, b) - \tfrac{1}{2}A_{ij}H_{jj}(a, cb).\end{aligned}$$

In (36), setting $c = 1$, $b = 1$, $i = j$ we get

$$H_{ii}(1, a) = 0, \tag{37}$$

and setting $a = 1$ and using (37), we have

$$H_{ij}(c, b) = \tfrac{1}{2}A_{ij}H_{jj}(c, b). \tag{38}$$

Also, using (34) twice,

$$H_{ij}(c, b) = -H_{ji}(b, c) = -\tfrac{1}{2}A_{ji}H_{ii}(b, c) = \tfrac{1}{2}A_{ji}H_{ii}(c, b).$$

Therefore

$$A_{ij}H_{jj}(c, b) = A_{ji}H_{ii}(c, b) \quad \text{for all } b, c \in R,\ 1 \le i, j \le l. \tag{39}$$

Suppose that $H_{j_0j_0}(c, b) \neq 0$ for some j_0 and some $c, b \in R$. Since $A = (A_{ij})$ is indecomposable, for any j, there exist $j_1, \dots, j_m$ where $j_m = j$ so that $A_{j_0j_1}A_{j_1j_2}\cdots A_{j_{m-1}j_m} \neq 0$. Iterating (39), we get $H_{jj}(c, b) = \varepsilon_j H_{j_0j_0}(c, b)$ for some $\varepsilon_j \in k$, $\varepsilon_j \neq 0$. Substitution into (39) yields $A_{ij}\varepsilon_j H_{j_0j_0}(c, b) = A_{ji}\varepsilon_i H_{j_0j_0}(c, b)$. As $H_{j_0j_0}(c, b) \neq 0$ it follows that $A_{ij}\varepsilon_j = A_{ji}\varepsilon_i$. This contradicts the fact that A is nonsymmetrizable.

We have shown that $H_{jj}(c, b) = 0$ for all $c, b \in R$, $1 \le j \le l$. Then (32) follows from (38) and (33) follows from (35).

Step 4. By Theorem 1, there exists a unique homomorphism τ from $g_R(A)$ to L so that $\tau(e_i(a)) = \tilde{e}_i(a)$, $\tau(f_i(a)) = \tilde{f}_i(a)$, and $\tau(h_i(a)) = \tilde{h}_i(a)$. Evidently, $\pi \circ \tau = id_{g_R(A)}$ which says that the original sequence splits.

Combining Theorem 2 with Haddi's result, we immediately have

Corollary 1. *If A is an indecomposable GCM, and if R satisfies $\Omega_R^1 \neq dR$ (for instance, $R = k[t, t^{-1}]$), then $H_2(g_R(A)) \neq 0$ if and only if A is symmetrizable.*

4. Remarks

Remark 1. Theorem 1 has already been worked out by Kassel [7] when A is of finite type. Our proof is more direct.

Remark 2. By further analyzing the proof of Theorem 2, letting $i = j$ in (36), we get

$$H_{ii}(c, ab) = H_{ii}(ca, b) - H_{ii}(a, cb).$$

From (34) and (37), we have

$$H_{ii}(1, a) = H_{ii}(a, 1) = 0,$$
$$H_{ii}(a, b) + H_{ii}(b, a) = 0,$$
$$H_{ii}(c, ab) + H_{ii}(a, bc) + H_{ii}(b, ca) = 0.$$

We thus get another proof for the fact that $H_2(g_R(A)) = \Omega_R^1/dR$ when A is an indecomposable and symmetrizable nonaffine GCM.

Remark 3. If the Lie algebra $g(A)$ is replaced by the Lie algebra $\hat{g}(A)$ which is defined by the relations: $[h_i, h_j] = 0$, $[h_i, e_j] = A_{ij}e_j$, $[h_i, f_j] = -A_{ij}f_j$, $[e_i, f_j] = \delta_{ij}h_j$, see [2, 6], then our methods also yield that $R \otimes_k \hat{g}(A)$ is centrally closed when A is indecomposable and nonsymmetrizable. Also, $H_2(R \otimes_k \hat{g}(A)) = \Omega_R^1/dR$ if A is an indecomposable and symmetrizable nonaffine GCM.

Acknowledgment

I would like to express my sincere gratitude to my supervisor, Professor S. Berman, for his constantly stimulating advice. Also, my thanks go to Y. Krylyuk for his helpful conversations.

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