PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 1, May 1994

A REMARK ON THE SPECTRAL SYNTHESIS PROPERTY FOR HYPERSURFACES OF R^n

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(Communicated by J. Marshall Ash)

ABSTRACT. Let M be an $(n-1)$-dimensional manifold in R^n with constant relative nullity. Using an estimate established in an earlier work of the author (Canad. Math. Bull. **36** (1993), 64–73), we present a greatly simplified proof of Müller's result on the weak spectral synthesis property of M.

1. INTRODUCTION

Let $S(R^n)$ be the space of Schwartz class functions and $S'(R^n)$ be the dual space of $S(R^n)$. Let $FL^\infty(R^n) = \{T \in S'(R^n),\ \widehat{T} \in L^\infty\}$. For a compact subset E of R^n and a positive integers m satisfying $m \geq n/2+1$, denote

$$\begin{aligned} I(E) &= \{f \in FL^1(R^n),\ f(E) = 0\}, \\ J(E) &= \{f \in C_0^m(R^n),\ f(E) = 0\}, \\ K(E) &= \{f \in C_0^\infty(R^n),\ f(E) = 0\}. \end{aligned}$$

Obviously, $\overline{K(E)} \subset \overline{J(E)} \subset I(E)$ in FL^1 norm.

If $\overline{K(E)} = I(E)$, then we say that E is of weak spectral synthesis.

If $\overline{J(E)} = I(E)$, then we say that E is of m-spectral synthesis.

Roughly speaking, the weak spectral synthesis property is a suitable concept for a C^∞ manifold, while the m-spectral synthesis property is a suitable concept for a C^m manifold. There is another concept, the spectral synthesis, which is a stronger property than the weak spectral synthesis. It can be proved that for a plane curve E and $m \geq 2$, the above three spectral synthesis properties are actually equivalent. We refer the interested reader to Domar's survey paper [3] for more information on the spectral synthesis and weak spectral synthesis and to [5] for the background of the m-spectral synthesis.

The C^∞ (or C^m) smoothness of E alone is not sufficient to imply the weak spectral synthesis property (or m-spectral synthesis property), as Domar's counterexample shows [4]. We need some curvature conditions.

Definition 1.1. Let B_δ^{n-1} be the open ball in R^{n-1} with radius δ and center at the origin, and let $F = \{(x, \psi(x));\ x \in B_\delta^{n-1}\}$ be a C^m $(m \geq n+1)$

Received by the editors August 27, 1992.
1991 *Mathematics Subject Classification.* Primary 43A15.

hypersurface of R^n. If the Hessian matrix of ψ, $(\partial^2\psi/\partial x_i \partial x_j)$, has constant rank $n-1-\nu$ on B_δ^{n-1}, $0 \le \nu \le n-1$, then we say that F has constant relative nullity ν. A C^m $(n-1)$-dimensional submanifold M of R^n is said to have constant relative nullity ν if every localization F of M has constant relative nullity ν.

Note that a manifold with constant relative nullity 0 is just a manifold with nonzero Gaussian curvature. For general n, the strongest result about the m-spectral synthesis property had been obtained by Müller [9, Theorem 2.2], who, in our context, proved

Müller's result. *Let M be a C^m $(n-1)$-dimensional manifold of R^n, $m \ge 2n+2$, with constant relative nullity ν, $1 \le \nu \le n-1$. If E is a compact subset of M with the restricted cone property* (*see Definition* 2.2), *then E is of m-spectral synthesis.*

To prove his result, Müller used Domar's ideas in [2] and [3], which makes the assumption $m \ge 2n+2$ imperative. However, Domar's result in [1] for the case $n=2$ indicates that the smoothness degree of the manifold M should be close to the space dimension n for large n.

Surprisingly enough, we find that an estimate obtained in the proof of Theorem 3 in [6] enables us to present a greatly simplified proof of Müller's result and at the same time to improve his result by reducing the smoothness assumption on the manifold M. The following is the main result of this note.

Theorem 1. *Let M be a C^m $(n-1)$-dimensional manifold of R^n with $m \ge n+5$ and with constant relative nullity ν, $1 \le \nu \le n-1$. If E is a compact subset of M with the restricted cone property, then E is of m-spectral synthesis.*

2. Notation and lemmas

Let us first recall the following definitions (see [2] or [9]).

Definition 2.1. A compact subset G of R^k is said to have the *restricted cone property at a point* $y_0 \in R^k$ if there exists a neighborhood V_0 of y_0 and a cone $K = \{y \in R^k : (1-\sigma)\|y\| \le y \cdot y_1 \le \sigma\}$, where $0 < \sigma < 1$, $y_1 \in R^k$, $\|y_1\| = 1$, such that $G \cap V_0 - K \subset F$.

Definition 2.2. A compact subset E of M is said to have the *restricted cone property* if for every $m \in M$ and every sufficiently small neighborhood V of M in R^k the orthogonal projection of $E \cap V$ to the tangent plane at m has the restricted cone property at m.

Definition 2.3. Let $x = (x_1, x_2, \dots, x_l, \dots, x_{n-1}) \in R^l \times R^{n-1-l}$. For a function $\psi(x)$, denote by $(H\psi)_l$ the matrix

$$\begin{pmatrix} \dfrac{\partial^2\psi}{\partial x_1 \partial x_1} & \cdots & \dfrac{\partial^2\psi}{\partial x_1 \partial x_1} \\ \vdots & & \vdots \\ \dfrac{\partial^2\psi}{\partial x_l \partial x_1} & \cdots & \dfrac{\partial^2\psi}{\partial x_l \partial x_l} \end{pmatrix}.$$

Note that $(H\psi)_{n-1}$ is just the Hessian matrix of ψ.

The proof of the following lemma is an easy exercise in linear algebra.

Lemma 2.1. *Let $\psi(x)$ be as in Definition 1.1 and let*

$$b_h(x, y) = (\psi(x - hy) - \psi(x))/h, \qquad (x, y) \in R^{n-1} \times R^{n-1},$$

where h is a small positive real number. Then there is a small δ' such that

(1) *If $(H\psi)_{n-1}$ has constant rank k in B_δ^{n-1}, then $(Hb_h)_{2(n-1)}$ has constant rank $2k$ in $B_{\delta'}^{n-1} \times B_{\delta'}^{n-1}$.*

(2) *If the determinant of $(H\psi)_k$ is away from zero for all $x \in B_\delta^{n-1}$, then the determinant of $(Hb_h)_{2k}$ is away from zero for all $(x, y) \in B_{\delta'}^{n-1} \times B_{\delta'}^{n-1}$ and all small h, where we arrange (x, y) as $(x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k, x_{k+1}, \dots, x_{n-1}, y_{k+1}, \dots, y_{n-1})$.*

Let $u = (u_1, u_2, \dots, u_k, \dots, u_p) \in R^p$ and let $G = \{(u, b(u)),\ u \in B_\delta^p\}$, where $b(u)$ is a $C^{[p/2]+5}$ real-valued function on B_δ^p such that the Hessian matrix of b has constant rank k. Without loss of generality, we may assume that the determinant of $(Hb)_k$ is away from zero on B_δ^p. Let $\alpha(u) \in C_0^{[p/2]+3}(B_\delta^p)$ and let $s^1 \in R^k$, $s^2 \in R^{p-k}$, and $\zeta \in R$. Also let $u^1 = (u_1, \dots, u_k)$, $u^2 = (u_{k+1}, \dots, u_p)$.

The proof of the following lemma needs only minor modification of the proof for Theorem 3 in [6]. For completeness, we give the proof here.

Lemma 2.2. *Let the manifold G be as above and let $S \in S'(R^{p+1})$ be given by the formula*

$$\langle S, f\rangle = \int_{B_\delta^p} f(u, b(u))\alpha(u)\, du$$

so that $\operatorname{supp} S \subset G$ and

$$\widehat{S}(s^1, s^2, \zeta) = C \int_{B_\delta^p} e^{-i(s^1, s^2, \zeta)\cdot(u^1, u^2, b(u^1, u^2))}\alpha(u^1, u^2)\, du^1\, du^2.$$

Then, we have

$$|\widehat{S}(s^1, s^2, \zeta)| \le C(1 + |s^1| + |\zeta|)^{-k/2}(1 + |(s^2 - t(s^1, \zeta)|)^{-((p-k)/2+1)},$$

where $t(s^1, \zeta)$ is a measurable function of (s^1, ζ) taking values in R^{p-k}.

To prove Lemma 2.2, we need

Hartman's Lemma ([7]; cf. [9, Lemma 5.2]). *Let M be a $C^{[p/2]+5}$ p-dimensional submanifold of R^{p+1} with constant relative nullity $p - k$, $0 \le k \le p$. Then for each $m_0 \in M$, there exists a bijective affine-linear transform τ_{m_0} of R^{p+1} such that at $\tau_{m_0}(m_0)$ the manifold $M^\tau = \tau_{m_0}(M)$ has a chart (X, Ω) with the following properties:*

(i) *$\Omega = B_{2\delta}^k \times B_\delta^{p-k}$, where δ is a small positive number.*

(ii) *For $v = (v', v'') \in \Omega$, $X(v) \in R^{p+1} = R^k \times R^{p-k} \times R$ such that*

$$X(v', v'') = (a(v') \cdot v'' + b(v'), v'', c(v') \cdot v'' + d(v')),$$

where a is a smooth matrix-valued function, b, c are smooth vector-valued functions, and d is a smooth scalar function.

(iii) *For each $v_0' \in B_{2\delta}^k$, the space of vectors normal to M^τ at a point $m' \in M' = \{X(v_0', v''): v'' \in B_\delta^{p-k}\}$ is independent of m'.*

(iv) *Let* $\Gamma(v', v'') = (a(v') \cdot v'' + b(v'), v'')$. *Then* Γ *is a* $C^{[p/2]+5}$ *diffeomorphism from* $\Omega = B^k_{2\delta} \times B^{p-k}_\delta$ *onto* $\Gamma(\Omega)$ *with* $\Gamma(0, v'') = (0, v'')$. *Let* $x = (x', x'') \in \Gamma(\Omega)$. *Then* $x'' = v''$.

(v) *If we define* $\psi(x) \in C^{[p/2]+5}(\Gamma(\Omega))$ *by* $\psi \circ \Gamma(v', v'') = c(v') \cdot v'' + d(v')$, *then* $\nabla\psi(0, x'') = 0$ *for all* $x'' \in B^\nu_\delta$, *and* $D^2\psi(0, 0)$ *is a diagonal matrix with real entries such that* $(D^2\psi(0, 0))_{i,i} = k_i \neq 0$ *for* $i = 1, \dots, k$, $(D^2\psi(0, 0))_{i,i} = 0$ *for* $i = k+1, \dots, p$. *And there exists* $\delta_1 > 0$ *such that* $\det[(\partial^2\psi/\partial x_i \partial x_j)|_{i,j=1}^{i,j=k}] \geq \delta_1 > 0$ *for all* $(x', x'') \in \Omega$.

Proof of Lemma 2.2. By taking δ smaller if necessary and by a bijective affine-linear transform of R^{p+1}, we may assume that G has the form (X, Ω) with the properties stated in Hartman's lemma. Also we may assume that the density function $\alpha(u) = A(v', v'')$ has the property that $A(v', v'') \in C_0^{[p/2]+5}(B^k_\delta \times B^{p-k}_\delta)$. For $(s^1, \zeta, s^2) \in R^k \times R \times R^{p-k} = R^{p+1}$, we have

$$\widehat{S}(s_1, s^2, \zeta) = \int_{B^{p-k}_\delta} \int_{B^k_{2\delta}} e^{-is^1\cdot(a(v')\cdot v'' + b(v')) + \zeta(c(v')\cdot v'' + d(v')) + s^2\cdot v'']} \cdot A(v', v'')\, dv'\, dv''. \tag{1}$$

For each $s^0 = (s^1, \zeta)/|(s^1, \zeta)| \in S^k$, the unit sphere in R^{k+1}, denote

$$g_{s^0, v''}(v') = s^0 \cdot (a(v') \cdot v'' + b(v'), c(v') \cdot v'' + d(v')).$$

Then (iii) of Hartman's lemma implies that the set $\{v' \in B^{n-1-\nu}_{2\delta}, \nabla g_{\eta^0, v''}(v') = 0\}$ is independent of v''. Also from (v) of Hartman's lemma, we see that for each $v'' \in B^{p-k}_\delta$, the k-dimensional submanifold $G_{v''} = \{(a(v') \cdot v'' + b(v'), c(v') \cdot v'' + d(v')), v' \in B^k_{2\delta}\}$ is $C^{[p/2]+5}$ smooth and the Gaussian curvature $k_{v''}(v')$ of $G_{v''}$ is away from zero, uniformly for $(v', v'') \in \Omega$. It follows that for δ small, we may assume that there exists $\varepsilon > 0$ such that for each $s^0 \in S^k$, either

$$|\nabla g_{s^0, v''}(v')| \geq \varepsilon \quad \text{uniformly for } (v', v'') \in B^k_\delta \times B^{p-k}_\delta \tag{2}$$

or there exist one and only one $v'_0 \in B^k_{2\delta}$ such that

$$\nabla g_{s^0, v''}(v'_0) = 0, \tag{3}$$

where v'_0 is independent of $v'' \in B^{p-k}_\delta$ and is uniquely determined by s^0 and hence by (s^1, ζ).

For case (2), integration by parts $[k/2]+1$ times for v', then $[(p-k)/2]+2$ times for v'' yields our Lemma 2.2. So it remains to consider case (3). Here the idea is to use the stationary phase method. Let

$$I(s^1, \zeta, v'') = \int_{B^k_{2\delta}} e^{-i[s^1\cdot(a(v')\cdot v'' + b(v')) + \zeta(c(v')\cdot v'' + d(v'))]} A(v', v'')\, dv'.$$

Then as $|s^1| + |\zeta| \to \infty$, $I(s^1, \zeta, v'') = P(s^1, \zeta, v'') + E(s^1, \zeta, v'')$, where P is the principal term of I and E is the error term. Let

$$P(s^1, \zeta, s^2) = \int_{B^{p-k}_\delta} P(s^1, \zeta, v'') e^{-is^2\cdot v''}\, dv'',$$

$$E(s^1, \zeta, s^2) = \int_{B^{p-k}_\delta} E(s^1, \zeta, v'') e^{-is^2\cdot v''}\, dv''.$$

The formula for $P(s^1, \zeta, v'')$ is well known (cf. [8, p. 331]). As $|s^1|+|\zeta| \to \infty$, $P(s^1, \zeta, v'')$ is

$$(4)\qquad \begin{aligned} &C(v_0', v'')A(v_0', v'')|k_{v''}(v_0')|^{-1/2} \\ &\quad \cdot e^{-i[s^1\cdot(a(v_0')\cdot v''+b(v_0'))+\zeta(c(v_0')\cdot v''+d(v_0'))]}(|s^1|+|\zeta|)^{-k/2}, \end{aligned}$$

where $C(v_0', v'')$ is a constant, uniformly bounded for all $v_0' \in B_{2\delta}^k$ and $v'' \in B_\delta^{p-k}$. Thus

(5)

$$\begin{aligned} P(s^1, \zeta, s^2) &= C(|s^1|+|\zeta|)^{-k/2} \\ &\quad \cdot \int_{B_\delta^{p-k}} e^{-i[s^1\cdot(a(v_0')\cdot v''+b(v_0'))+\zeta(c(v_0')\cdot v''+d(v_0'))+s^2\cdot v'']}B(v_0', v'')\,dv'', \end{aligned}$$

where v_0' is a $C^{[p/2]+4}$ smooth function of s^0, $B(v_0', v'')$ is a $C^{[p/2]+4}$ smooth function of (v_0', v'').

For small $|s^1|+|\zeta|$, from the definition of $P(s^1, \zeta, v'')$ and $P(s^1, \zeta, s^2)$, it is easy to see

$$(6)\qquad |P(s^1, \zeta, s^2)| \le C(1+|\xi|)^{-((p-k)/2+1)}.$$

Hence from (5) and (6), we have

$$(7)\qquad |P(s^1, \zeta, s^2)| \le C(1+|s^1|+|\zeta|)^{-k/2}(1+|(s^2-t(s^1, \zeta))|)^{-((p-k)/2+1)},$$

where $t(s^1, \zeta)$ is a measurable function of (s^1, ζ), taking values in R^{p-k}.

Moreover a detailed calculation in the stationary phase method yields

$$(8)\qquad |E(s^1, \zeta, s^2)| \le C(1+|s^1|+|\zeta|)^{-(k/2+1)}(1+|(s^2-t(s^1, \zeta))|)^{-((p-k)/2+1)}.$$

Note that in (4), (6), (7), and (8), we used the smoothness assumption of the manifold G. From (7) and (8), we have

$$(9)\qquad |\widehat{S}(s^1, s^2, \zeta)| \le C(1+|s^1|+|\zeta|)^{-k/2}(1+|(s^2-t(s^1, \zeta))|)^{-((p-k)/2+1)}.$$

The proof of Lemma 2.2 is complete.

Let F be as in Definition 1.1 and let $T \in FL^\infty(R^n)$ such that $\operatorname{supp} T \subset F$ and T vanishes on $J(F)$. Notice that if $f \in C_0^{n+1}(R^n)$, then $\hat{f} \in L^1(R^n)$. So the pair $\langle T, f\rangle$ makes sense. Following Domar, we now construct a family of *good measures* $\{T_h\}$ on F for all small positive h as follows. Let

$$\begin{aligned} \pi&: R^n \to R^{n-1} \quad \text{given by } (x, z) \to x, \\ \beta&: B_\delta^{n-1} \to R^n \quad \text{given by } x \to (x, \psi(x)). \end{aligned}$$

We first define a distribution $\Sigma \in S'(R^{n-1})$ by

$$\langle \Sigma, g\rangle = \langle T, g\circ\pi\rangle \text{ for } g \in S(R^{n-1}).$$

This makes sense since $\operatorname{supp}(T)$ is compact. From the construction of Σ, it is obvious that $\operatorname{supp}(\Sigma) \subset B_\delta^{n-1}$. It follows that we can find $\gamma(x) \in C_0^\infty(B_\delta^{n-1})$ such that $\Sigma = \gamma\Sigma$. Let $\phi(x) \in C_0^\infty(B_\delta^{n-1})$ and $\int_{R^{n-1}} \phi(x)\,dx = 1$. Denote $\phi_h(x) = \phi(x/h)h^{n-1}$ and $\check{\phi}_h(x) = \phi_h(-x)$.

For $f \in C_0^{n+2}(B_\delta^n)$, we let $f_\beta(x) = f\circ\beta(x) = f(x, \psi(x))$ for $x \in B_\delta^n$.

Now we define $T_h \in S'(R^n)$ for $0 < h < \frac{1}{2}\operatorname{dist}(\partial U, \operatorname{supp}(\Sigma))$ by

$$\langle T_h, f\rangle = \langle \Sigma * \check{\phi}_h, f \circ \beta\rangle \quad \text{for } f \in S(R^n).$$

Since $\operatorname{supp}\Sigma * \check{\phi}_h \subset B_\delta^{n-1}$ for all small h, it is easy to check that T_h is well defined and T_h is a mass measure on F with a $C_0^{m-1}(F)$ density function.

3. Proof of Theorem 1

Proof of Theorem 1. Let T and T_h be as in §2. Following a standard localization argument (see [1, p. 31]) and the definition of the restricted cone property, it is easy to see that Theorem 1 will be proved if we can show the inequality

$$\|\widehat{T}_h\|_\infty \leq C\|\widehat{T}\|_\infty \tag{10}$$

where C is independent of all small h.

For $(\eta, \xi) \in R^{n-1} \times R$ and $(x, x_n) \in R^{n-1} \times R$, let $X(x, x_n) = e^{i(\eta\cdot x + \xi x_n)}$. Then from the construction of T_h, we have

$$\begin{aligned}\widehat{T}_h(\eta, \xi) = \langle T_h, X\rangle &= \langle \gamma\Sigma, X \circ \beta * \phi_h\rangle \\ &= \langle T, \gamma e^{i\xi x_n} e^{-i\xi\psi(x)}(X \circ \beta) * \phi_h\rangle.\end{aligned}$$

Here in the last identity, we used the definition of Σ and the assumption that T vanishes on $J(F)$ and ψ is C^{n+5} smooth. It follows that

$$\begin{aligned}\widehat{T}_h(\eta, \xi) &= \left\langle T, e^{i\xi x_n}\gamma(x)e^{-i\xi\psi(x)} \int_{R^{n-1}} e^{i(\langle \eta, x-y\rangle + \xi\psi(x-y))}\phi_h(y)\,dy\right\rangle \\ &= \left\langle T, e^{i\xi x_n}\gamma(x)e^{-i\xi\psi(x)} \int_{R^{n-1}} e^{i(\langle \eta, x-hy\rangle + \xi\psi(x-hy))}\phi(y)\,dy\right\rangle \\ &= \left\langle T, e^{i(\langle \eta, x\rangle + \xi x_n}\gamma(x)e^{-i\xi\psi(x)} \int_{R^{n-1}} e^{i(\langle \eta, -hy\rangle + \xi\psi(x-hy))}\phi(y)\,dy\right\rangle \\ &= \langle T, e^{i(\langle \eta, x\rangle + \xi x_n} g_{n,\xi}(x)\rangle = C\left\langle T, e^{i(\langle \eta, x\rangle + \xi x_n} \int_{R^{n-1}} e^{iw\cdot x}\hat{g}_{\eta,\xi}(w)\,dw\right\rangle \\ &= C\left\langle T, \int_{R^{n-1}} e^{i(\langle \eta + w, x\rangle + \xi x_n}\hat{g}_{n,\xi}(w)\,dw\right\rangle \\ &= C\int_{R^{n-1}} \widehat{T}(\eta + w, \xi)\hat{g}_{\eta,\xi}(w)\,dw\end{aligned}$$

where $g_{\eta,\xi,h}(x) = \gamma(x)e^{-i\xi\psi(x)} \int_{R^{n-1}} e^{i(\langle \eta, -hy\rangle + \xi\psi(x-hy))}\phi(y)\,dy$.

So

$$\hat{g}_{\eta,\xi,h}(w) = \int_{R^{n-1}}\int_{R^{n-1}} e^{i(w\cdot x + h\eta\cdot y + h\xi((\psi(x-hy)-\psi(x))/h)}\phi(y)\gamma(x)\,dy\,dx.$$

Denote $u = (x, y) = (x_1, \dots, x_{n-1-\nu}, y_1, \dots, y_{n-1-\nu}, x_{n-\nu}, \dots, x_{n-1}, y_{n-\nu}, \dots, y_{n-1})$, $w^1 = (w_1, \dots, w_{n-1-\nu})$, $w^2 = (w_{n-\nu}, \dots, w_{n-1})$, $\eta^1 = (\eta_1, \dots, \eta_{n-1-\nu})$, $\eta^2 = (\eta_{n-\nu}, \dots, \eta_{n-1})$. Then Lemma 2.1 enables us to apply Lemma 2.2 with $p = 2(n-1)$, $k = 2(n-1-\nu)$, $s^1 = (w^1, h\eta^1)$, $s^2 = (w^2, h\eta^2)$, and $\zeta = h\xi$ to obtain

$$\begin{aligned}|\hat{g}_{\eta,\xi,h}(w)| \leq\ & C(1 + |w^1| + |h\eta^1| + |h\xi|)^{-(n-1-\nu)} \\ & \cdot (1 + |(w^2, h\eta^2) - t(s^1, h\xi)|)^{-(\nu+1)}\end{aligned} \tag{11}$$

where C is independent of η, ξ, w, and h.

We want to show that

$$|\hat{g}_{\eta,\xi,h}(w)| \le C(1+|h\xi|)^{-(n-1-\nu)}\left(\frac{|w_1|}{1+|h\xi|}\right)^{-(n-\nu)} \cdot (1+|(w^2, h\eta^2)-t(s^1, h\xi)|)^{-(\nu+1)}. \tag{12}$$

Let

$$M = 2\max\left\{\left|\nabla_{x_1}\left(\frac{\psi(x-hy)-\psi(x)}{h}\right)\right| ; (x, y) \in B_\delta^{n-1}\times B_\delta^{n-1}, \text{ all small } h\right\}.$$

If $|w^1| \le M(1+|h\xi|)$, then (11) gives (12) immediately. For $|w^1| \ge M(1+|h\xi|)$, we denote

$$a_{w,\xi,h}(x, y) = D_{w^l}\left[i\frac{1}{D_{w^1}(i\frac{w}{|w^1|}\cdot x + h\frac{\eta}{|w^1|}\cdot y + h\frac{\xi}{|w^1|}(\frac{\psi(x-hy)-\psi(x)}{h}))}\phi(y)\gamma(x)\right],$$

where D_{w^1} is the derivative operator in the direction w^1.

Then it is easy to see that $a_{w,\xi,h}(x, y) \in C_0^{n+2}(B_\delta^{n-1}\times B_\delta^{n-1})$ such that $\|a_{w,\xi,h}(\cdot, \cdot)\|_{C^{n+2}(B_\delta^{n-1}\times B_\delta^{n-1})} \le K$, K independent of w, ξ, and h.

Now by integration by parts for the variable $x^1 = (x_1, \dots, x_{n-1-\nu})$ in the direction w^1, we get

$$\hat{g}_{\eta,\xi,h}(w) = \frac{1}{|w^1|}\int_{R^{n-1}}\int_{R^{n-1}} e^{i(w\cdot x + h\eta\cdot y + h\xi((\psi(x-hy)-\psi(x))/h))} a_{w,\xi,h}(x, y)\,dy\,dx.$$

Since $|w^1|^{-1} \le (w^1/(1+|\xi|))^{-1}$, we see that (12) follows from Lemma 2.2 again with the function a there replaced by $a_{w,\xi,h}$ here. Note that for $|w| \le 1$, $|\hat{g}_{\eta,\xi,h}(w)| \le C$, uniformly for η, ξ, and h. It follows from (12) that

$$\int_{R^{n-1-\nu}}\int_{R^\nu} |\hat{g}_{\eta,\xi,h}(w^1, w^2)|\,dw^2\,dw^1 \le C$$

with C independent of η, ξ, and h and hence inequality (10) holds. This proves Theorem 1.

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