DERIVATIONS IN PRIME NEAR-RINGS

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ABSTRACT. Let N be a prime near-ring with center Z . The purpose of this paper is to study derivations on N . We show two main results: (1) Let N be 2-torsion-free, and let D_1 and D_2 be derivations on N such that D_1D_2 is also a derivation. Then either D_1 or D_2 is zero if and only if $[D_1(x), D_2(y)] = 0$ for all $x, y \in N$. (2) Let n be an integer ≥ 1 , N be $n!$ -torsion-free, and D a derivation with $D^n(N) = \{0\}$. Then $D(Z) = \{0\}$.

Throughout this paper N always stands for a zero-symmetric left near-ring. An additive endomorphism D of N is called a derivation on N if $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$. According to [1], a near-ring N is said to be prime if $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$.

As the addition of a near-ring is not necessarily commutative, the following Proposition 1 has its own significance.

Proposition 1. *Let D be an arbitrary additive endomorphism of N . Then $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$ if and only if $D(xy) = D(x)y + xD(y)$ for all $x, y \in N$. Therefore D is a derivation if and only if $D(xy) = D(x)y + xD(y)$.*

Proof. Suppose $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$. Since $x(y + y) = xy + xy$ and

$D(x(y + y)) = xD(y + y) + D(x)(y + y) = xD(y) + xD(y) + D(x)y + D(x)y$
 and

$$D(xy + xy) = D(xy) + D(xy) = xD(y) + D(x)y + xD(y) + D(x)y,$$

we get $xD(y) + D(x)y = D(x)y + xD(y)$, so $D(xy) = D(x)y + xD(y)$.

The converse is proved in a similar way.

Lemma 1. *Let D be an arbitrary derivation on N . Then N satisfies the following partial distributive laws (for $x, y, z \in N$):*

- (i) $(xD(y) + D(x)y)z = xD(y)z + D(x)yz$;
- (ii) $(D(x)y + xD(y))z = D(x)yz + xD(y)z$.

Proof. (i) was proved in [1], so we only need to prove (ii). From the associative law and Proposition 1 we have

$$D((xy)z) = D(xy)z + xyD(z) = (D(x)y + xD(y))z + xyD(z)$$

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and

$$\begin{aligned} D(x(yz)) &= D(x)yz + xD(yz) = D(x)yz + x(D(y)z + yD(z)) \\ &= D(x)yz + xD(y)z + xyD(z). \end{aligned}$$

Comparing the two expressions we obtain

$$(D(x)y + xD(y))z = D(x)yz + xD(y)z.$$

Now we prove our first main result, which extends a famous theorem on rings of Posner [2] to near-rings.

Theorem 1. *Let N be a 2-torsion-free prime near-ring, and let D_1 and D_2 be derivations on N such that D_1D_2 is also a derivation. Then the following two conditions are equivalent:*

- (i) *either $D_1 = 0$ or $D_2 = 0$;*
- (ii) *$[D_1(x), D_2(y)] = 0$ for all $x, y \in N$.*

Proof. (i) $\Rightarrow$ (ii) is obvious. We only prove (ii) $\Rightarrow$ (i). Noting that D_1D_2 is a derivation, we have

$$D_1D_2(xy) = xD_1D_2(y) + D_1D_2(x)y.$$

On the other hand, D_1 and D_2 are both derivations, so

$$\begin{aligned} D_1D_2(xy) &= D_1(D_2(xy)) = D_1(xD_2(y) + D_2(x)y) \\ &= D_1(xD_2(y)) + D_1(D_2(x)y) \\ &= xD_1D_2(y) + D_1(x)D_2(y) + D_2(x)D_1(y) + D_1D_2(x)y. \end{aligned}$$

The above two expressions for $D_1D_2(xy)$ yield

$$(1) \quad D_1(x)D_2(y) + D_2(x)D_1(y) = 0 \quad \text{for all } x, y \in N.$$

Replacing x by $xD_2(z)$ in (1), by using Proposition 1 and Lemma 1 we have

$$\begin{aligned} 0 &= D_1(xD_2(z))D_2(y) + D_2(xD_2(z))D_1(y) \\ &= (D_1(x)D_2(z) + xD_1D_2(z))D_2(y) + (xD_2^2(z) + D_2(x)D_2(z))D_1(y) \\ &= D_1(x)D_2(z)D_2(y) + xD_1D_2(z)D_2(y) + xD_2^2(z)D_1(y) + D_2(x)D_2(z)D_1(y) \\ &= D_1(x)D_2(z)D_2(y) + x(D_1D_2(z)D_2(y) + D_2^2(z)D_1(y)) + D_2(x)D_2(z)D_1(y). \end{aligned}$$

In this equality

$$x(D_1D_2(z)D_2(y) + D_2^2(z)D_1(y)) = 0$$

because the second factor $D_1D_2(z)D_2(y) + D_2^2(z)D_1(y) = 0$ by equality (1) with x replaced by $D_2(z)$. Thus we get

$$(2) \quad D_1(x)D_2(z)D_2(y) + D_2(x)D_2(z)D_1(y) = 0 \quad \text{for all } x, y, z \in N.$$

Replacing x and y by z in (1), respectively, we obtain

$$D_2(z)D_1(y) = -D_1(z)D_2(y)$$

and

$$D_1(x)D_2(z) = -D_2(x)D_1(z).$$

Since N is a zero-symmetric left near-ring, (2) becomes

$$\begin{aligned} 0 &= (-D_2(x)D_1(z))D_2(y) + D_2(x)(-D_1(z)D_2(y)) \\ &= D_2(x)(-D_1(z))D_2(y) + D_2(x)(-D_1(z)D_2(y)) \\ &= D_2(x)[(-D_1(z))D_2(y) - D_1(z)D_2(y)] \end{aligned}$$

for all $x, y, z \in N$. If $D_2 \neq 0$, by [1, Lemma 3] we have

$$(-D_1(z))D_2(y) - D_1(z)D_2(y) = 0,$$

that is,

$$(3) \quad D_1(z)D_2(y) = (-D_1(z))D_2(y) \quad \text{for all } y, z \in N.$$

However, by condition (ii) we have

$$\begin{aligned} (-D_1(z))D_2(y) &= D_1(-z)D_2(y) = D_2(y)D_1(-z) \\ &= D_2(y)(-D_1(z)) = -D_2(y)D_1(z) = -D_1(z)D_2(y), \end{aligned}$$

that is,

$$(4) \quad (-D_1(z))D_2(y) = -D_1(z)D_2(y) \quad \text{for all } y, z \in N.$$

From (3) and (4) we obtain

$$2D_1(z)D_2(y) = 0,$$

or $D_1(z)D_2(y) = 0$ since N is 2-torsion-free. Hence $D_1(z)D_2(N) = \{0\}$, but $D_2 \neq 0$ so $D_1(z) = 0$ for all $z \in N$. Thus $D_1 = 0$.

As a consequence of Theorem 1, we obtain

Corollary 1 [1]. *Let N be a 2-torsion-free prime near-ring, and let D be a derivation on N such that $D^2 = 0$. Then $D = 0$.*

Proof. It is clear that $D^2 = 0$ is a derivation on N , and we have

$$\begin{aligned} 0 &= D^2(xy) = D(xD(y) + D(x)y) = D(xD(y)) + D(D(x)y) \\ &= xD^2(y) + D(x)D(y) + D(x)D(y) + D^2(x)y = 2D(x)D(y), \end{aligned}$$

so $D(x)D(y) = 0$. Therefore $[D(x), D(y)] = 0$ for all $x, y \in N$. From Theorem 1, $D = 0$.

Using equality (1) in the proof of Theorem 1 we can prove the following interesting result.

Proposition 2. *Let N be a near-ring and D_1 and D_2 be derivations on N such that D_1D_2 is a derivation. Then D_2D_1 is also a derivation.*

Proof. Obviously D_2D_1 is an additive endomorphism of N . By equality (1) and Proposition 1 we have

$$\begin{aligned} D_2D_1(xy) &= D_2(D_1(x)y + xD_1(y)) = D_2(D_1(x)y) + D_2(xD_1(y)) \\ &= D_2D_1(x)y + (D_1(x)D_2(y) + D_2(x)D_1(y)) + xD_2D_1(y) \\ &= D_2D_1(x)y + xD_2D_1(y) \end{aligned}$$

for all $x, y \in N$. Thus D_2D_1 is a derivation by Proposition 1.

Corollary 1 leads us naturally to ask a question: let integer $n \geq 2$, and let N be an $n!$ -torsion-free prime near-ring. If D is a derivation on N such that $D^n(N) = \{0\}$, can we conclude $D(N) = \{0\}$?

The answer is negative even for rings. A simple counterexample due to Chung, Kobayashi, and Luh [3] is as follows: Let R be the ring of 2×2 matrices over $GF(P)$, where P is a prime integer greater than 3 and D be the inner derivation induced by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then N is $3!$ -torsion-free and $D^3(R) = \{0\}$, but $D(R) \neq \{0\}$.

Nevertheless we will show that in the case of near-ring $D(Z) = \{0\}$ where Z is the center of N .

In order to discuss the question we need to extend Leibniz' rule for derivations of rings to near-rings.

Proposition 3. *Leibniz' rule holds in near-rings, namely, for any integer $n \geq 2$ and any $x, y \in N$, it holds that*

$$\begin{aligned} D^n(xy) &= D^n(x)y + \binom{n}{1} D^{n-1}(x)D(y) + \cdots + \binom{n}{i} D^{n-i}(x)D^i(y) \\ &\quad + \cdots + \binom{n}{n-1} D(x)D^{n-1}(y) + xD^n(y). \end{aligned}$$

Proof. Using Proposition 1 and elementary facts about centralizers of elements in group, one can easily prove

$$D(x)y + nxD(y) = nxD(y) + D(x)y.$$

Further, we can prove

$$(5) \quad nD(x)y + nxD(y) = n(D(x)y + xD(y)) \quad \text{for all } x, y \in N.$$

Next we prove Leibniz' rule by induction on n . When $n = 2$ we have

$$\begin{aligned} D^2(xy) &= D(D(x)y + xD(y)) \\ &= D(D(x)y) + D(xD(y)) \\ &= D^2(x)y + D(x)D(y) + D(x)D(y) + xD^2(y) \\ &= D^2(x)y + 2D(x)D(y) + xD^2(y). \end{aligned}$$

Assume Leibniz' rule holds for $n - 1$. That is, if N is $(n - 1)!$ -torsion-free, then

$$\begin{aligned} D^{n-1}(xy) &= D^{n-1}(x)y + \cdots + \binom{n-1}{i-1} D^{n-i}(x)D^{i-1}(y) \\ &\quad + \binom{n-1}{i} D^{n-i-1}(x)D^i(y) + \cdots + xD^{n-1}(y). \end{aligned}$$

Since $n!$ -torsion-freeness implies $(n-1)!$ -torsion-freeness, by (5) we have

$$\begin{aligned}
 D^n(xy) &= D(D^{n-1}(xy)) \\
 &= D(D^{n-1}(x)y + \cdots + \binom{n-1}{i-1} D^{n-i}(x) D^{i-1}(y) \\
 &\quad + \binom{n-1}{i} D^{n-i-1}(x) D^i(y) + \cdots + x D^{n-1}(y)) \\
 &= D(D^{n-1}(x)y) + \cdots + \binom{n-1}{i-1} D(D^{n-i}(x) D^{i-1}(y)) \\
 &\quad + \binom{n-1}{i} D(D^{n-i-1}(x) D^i(y)) + \cdots + D(x D^{n-1}(y)) \\
 &= D^n(x)y + D^{n-1}(x)D(y) \\
 &\quad + \cdots + \binom{n-1}{i-1} (D^{n-i+1}(x) D^{i-1}(y) + D^{n-i}(x) D^i(y)) \\
 &\quad + \binom{n-1}{i} (D^{n-i}(x) D^i(y) + D^{n-i-1}(x) D^{i+1}(y)) \\
 &\quad + \cdots + D(x) D^{n-1}(y) + x D^n(y) \\
 &= D^n(x)y + \cdots + \binom{n-1}{i-1} D^{n-i+1}(x) D^{i-1}(y) + \binom{n-1}{i-1} D^{n-i}(x) D^i(y) \\
 &\quad + \binom{n-1}{i} D^{n-i}(x) D^i(y) + \binom{n-1}{i} D^{n-i-1}(x) D^{i+1}(y) + \cdots + x D^n(y) \\
 &= D^n(x)y + \cdots + \binom{n-1}{i-1} D^{n-i}(x) D^i(y) + \binom{n-1}{i} D^{n-i}(x) D^i(y) \\
 &\quad + \cdots + x D^n(y) \\
 &= D^n(x)y + \cdots + \left[\binom{n-1}{i-1} + \binom{n-1}{i} \right] D^{n-i}(x) D^i(y) \\
 &\quad + \cdots + x D^n(y) \\
 &= D^n(x)y + \cdots + \binom{n}{i} D^{n-i}(x) D^i(y) + \cdots + x D^n(y).
 \end{aligned}$$

The proof is completed.

Lemma 2. Let N be a near-ring with center Z , and let D be a derivation on N . Then $D(Z) \subseteq Z$.

Proof. From Proposition 1, for any $z \in Z$ and any $x \in N$ we have

$$xD(z) + zD(x) = xD(z) + D(x)z = D(xz) = D(zx) = D(z)x + zD(x).$$

It follows that $xD(z) = D(z)x$, that is, $D(z) \in Z$.

Lemma 3. Let $n \geq 2$, and let N be an $n!$ -torsion-free near-ring and D be a derivation with $D^n(N) = \{0\}$. Then for each $y \in N$, either $D(y) = 0$ or there exists $0 < k < n$ such that $D^k(y)$ is a nonzero divisor of zero.

Proof. Since $n!$ -torsion-freeness implies $(n-1)!$ -torsion-freeness, we may assume that $D^{n-1}(N) \neq \{0\}$, in which case we choose x_0 such that $D^{n-1}(x_0) \neq 0$. Assume $D(y) \neq 0$. Then there exists k with $0 < k < n$ for which $D^k(y) \neq 0$ and $D^{k+1}(y) = 0$.

Using Leibniz' rule we obtain

$$\begin{aligned} 0 &= D^n(x_0 D^{k-1}(y)) = D^n(x_0) D^{k-1}(y) + \binom{n}{1} D^{n-1}(x_0) D^k(y) \\ &\quad + \binom{n}{2} D^{n-2}(x_0) D^{k+1}(y) + \cdots \\ &= \binom{n}{1} D^{n-1}(x_0) D^k(y) = n D^{n-1}(x_0) D^k(y). \end{aligned}$$

We get $D^{n-1}(x_0) D^k(y) = 0$ since N is $n!$ -torsion-free. So $D^k(y)$ is a nonzero divisor of zero.

Now we can prove our second main theorem.

Theorem 2. *Let n be an integer ≥ 1 and N be a prime near-ring with center Z , and let N be $n!$ -torsion-free and D a derivation with $D^n(N) = \{0\}$. Then $D(Z) = \{0\}$.*

Proof. If $n = 1$, there is nothing to prove. If $n \geq 2$, suppose $D(Z) \neq \{0\}$. We choose $z \in Z$ such that $D(z) \neq 0$. By Lemmas 2 and 3, there exists a positive integer k such that $D^k(z)$ is a nonzero divisor of zero contained in Z . On the other hand, $D(z)$ could not be a divisor by [1, Lemma 3]. The contradiction proves that $D(Z) = \{0\}$.

Finally we drop the condition that N is prime to obtain the following

Theorem 3. *Let n be a positive integer and N be an $n!$ -torsion-free near-ring with no divisor of zero, then N admits no nonzero derivation D with $D^n = 0$.*

The proof is immediately obtained by Lemma 3.

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