KRULL DIMENSION OF MODULES OVER INVOLUTION RINGS

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ABSTRACT. The following question of Lanski is answered positively in the case when a ring R with involution $*$ is Noetherian with respect to two-sided $*$ -ideals. Let $\bar{R}$ be a ring with $*$ and invertible 2, and let $\bar{S}$ be the subring of R generated by the symmetric elements in R . Does any left R -module have the same Krull dimension when considered as an R -module and $\bar{S}$ -module?

Our starting point is a question implicit in a paper of Lanski [7], namely given a ring R with involution $*$, with $1/2 \in R$, and with $\bar{S}$ the subring generated by the symmetric elements in R , and given any left R -module M , does M have the same Krull dimension when considered as an R -module and as an $\bar{S}$ -module? The question is a very natural one, given developments in the theory of rings with involution in the 1970s, but is surprisingly hard to settle. Indeed Lanski [7, p. 398] frankly admits that he did not know the answer even in the case of an Artinian R -module M . To the best of our knowledge, this is still the case.

Let us review some of the positive results related to this question. In an earlier paper, Lanski [5] proved that if R is an Artinian ring then so too is $\bar{S}$ and R is a finitely generated $\bar{S}$ -module. In [6], Lanski continued his investigations and proved that if R is a Noetherian ring then R is a Noetherian $\bar{S}$ -module and the Krull dimensions of R as an R -module and as an $\bar{S}$ -module are the same. The first part of this theorem of Lanski was proved independently by Chuang and Lee [1].

In this note we shall prove that if R satisfies the ascending chain condition on two-sided $*$ -ideals then any left R -module has the same (dual) Krull dimension when viewed as an R -module and as an $\bar{S}$ -module. This theorem contains all the above results as special cases. As one might expect, the arguments in the cases of Krull dimension and dual Krull dimension are dual to each other in many places, but Lemma 7 and its “dual” Lemma 7* do differ in some important regards. To be careful and for the convenience of the reader, we give both arguments, the dual argument being given in parentheses.

Naturally we are disappointed not to be able to settle the general question one way or the other, but we present our results in the belief that the methods are interesting. Basically, in the spirit of Lanski’s work, we make certain reductions

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until we can exploit the fact that we know precisely what the involution $*$ must be. This approach does indicate in general terms what must be proved if the above question has a positive answer.

All rings in this paper will be assumed to be associative and all modules unital left modules. Given a ring R and an R -module M , the Krull dimension of M will be denoted $k_R(M)$ and the dual Krull dimension of M will be denoted $k_R^*(M)$. We start with two general observations. The first is probably well known but we include its proof for completeness.

Proposition 1. *Suppose that S is a subring of a ring R . Suppose there exists an ordinal $\alpha \geq 0$ and an R -module M such that $k_R(M) = \alpha$ (respectively, $k_R^*(M) = \alpha$) but $k_S(M) \neq \alpha$ ($k_S^*(M) \neq \alpha$), and suppose further that α is the least ordinal with this property. Then the module M contains an R -submodule N such that $k_S(N) \not\leq \alpha$ ($k_S^*(M/N) \not\leq \alpha$) and, for every R -submodule K of N , either $k_S(K) \leq \alpha$ or $k_S(N/K) < \alpha$ ($k_S^*(N/K) \leq \alpha$ or $k_S^*(K) < \alpha$).*

Proof. We prove the proposition in the Krull dimension case, the other case being proved by dualizing the argument.

Suppose the result (for Krull dimension) is false. Then M contains an R -submodule N_1 such that $k_S N_1 \not\leq \alpha$ and $k_S(M/N_1) \not\leq \alpha$. Note that, by the choice of α , $k_R(M/N_1) = \alpha$. Now, there exists an R -submodule N_2 of N_1 such that $k_S(N_2) \not\leq \alpha$ and $k_S(N_1/N_2) \not\leq \alpha$. Again, $k_R(N_1/N_2) = \alpha$. Repeating this argument, we obtain a chain of R -submodules $M = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$ such that $k_R(N_{i-1}/N_i) = \alpha$ for all $i \geq 1$, contradicting the fact that $k_R(M) = \alpha$.

Proposition 2. *Let S be a subring of a ring R and let I be an ideal of R such that $I \subseteq S$. Suppose that every (R/I) -module has the same (dual) Krull dimension when considered as an (R/I) -module and as an (S/I) -module. Then every R -module has the same (dual) Krull dimension as an R -module and as an S -module.*

Proof. Suppose that there exists an R -module M such that $k_R(M) = \alpha$ and $k_S(M) \neq \alpha$ (respectively, $k_R^*(M) = \alpha$ and $k_S^*(M) \neq \alpha$), for some ordinal $\alpha \geq 0$. Without loss of generality, we can suppose that the ordinal α is minimal with respect to this property. Proposition 1 allows us to assume that, for every R -submodule N of M , $k_S(N) \leq \alpha$ or $k_S(M/N) < \alpha$ (respectively, $k_S^*(N) < \alpha$ or $k_S^*(M/N) \leq \alpha$).

Since $k_S(M) \neq \alpha$ ($k_S^*(M) \neq \alpha$), there exists a chain

$$M = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \quad (0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots)$$

of S -submodules of M such that $k_S(N_{i-1}/N_i) \not\leq \alpha$ ($k_S^*(N_i/N_{i-1}) \not\leq \alpha$) for all $i \geq 1$. Let $N = RN_1$. Then

$$IN = IRN_1 = IN_1 \subseteq SN_1 \subseteq N_1.$$

Hence $k_S(M/IN) \not\leq \alpha$ ($k_S^*(N) \not\leq \alpha$). The assumption on submodules of M gives $k_S(IN) \leq \alpha$ ($k_S^*(M/IN) \leq \alpha$). By the hypothesis,

$$k_R(N/IN) = k_{R/I}(N/IN) = k_{S/I}(N/IN) = k_S(N/IN),$$

so that $k_S(N/IN) \leq \alpha$ ($k_S^*(N/IN) \leq \alpha$). Consequently, $k_S(N) \leq \alpha$ ($k_S^*(M/IN) \leq \alpha$). However, $N_1 \subseteq N$, so that $k_S(N_1) \leq \alpha$ (respectively, $IN \subseteq N_1$, so that $k_S^*(M/N_1) \leq \alpha$), a contradiction.

We shall require the following additional fact.

Proposition 3. (i) Let R be a ring such that $1/2 \in R$. Let σ be an automorphism of order 2 of R and let S denote the fixed subring of R with respect to σ , i.e., $S = R^\sigma = \{r \in R: \sigma(r) = r\}$. Then $k_R(M) = k_S(M)$ and $k_R^*(M) = k_S^*(M)$, for every R -module M .

(ii) Let T be a subring of a ring R and I a nilpotent ideal of R . Then $k_{(T+I)}(M) = k_T(M)$ and $k_{(T+I)}^*(M) = k_T^*(M)$, for every $(T+I)$ -module M .

Proof. In [2, Theorem 1] it was proved that if a ring R is graded by a finite group and R' denotes the identity component R_e of R then $k_R(M) = k_{R'}(M)$ and $k_R^*(M) = k_{R'}^*(M)$, for any R -module M .

Now to prove (i), it suffices to note that $R = S \oplus S'$, where $S' = \{r \in R: \sigma(r) = -r\}$, gives a C_2 -grading on R with identity component S .

To prove (ii), note that if $I^n = 0$, for some positive integer n , then the subring $A = T + Ig + I^2g^2 + \cdots + I^{n-1}g^{n-1}$ of the group ring $R[C_n]$, where $C_n = \langle g; g^n = 1 \rangle$, is a C_n -graded ring. The epimorphism $\varphi: A \rightarrow T + I$ defined by

$$\varphi(t + i_1g + \cdots + i_{n-1}g^{n-1}) = t + i_1 + \cdots + i_{n-1},$$

where $t \in T$, $i_j \in I$ ($1 \leq j \leq n-1$), makes every $(T+I)$ -module M an A -module such that $k_A(M) = k_{(T+I)}(M)$ and $k_A^*(M) = k_{(T+I)}^*(M)$.

Having completed these preliminaries we now turn to rings with involution.

The following notation will be used in the remainder of this paper: R is a ring with involution $*$ such that $1/2 \in R$, $S(R) = \{r \in R: r^* = r\}$ is the set of symmetric elements of R , and $\overline{S}(R)$ the subring of R generated by $S(R)$. For all x, y in R , as usual $[x, y]$ will denote the commutator $xy - yx$.

Lemma 4. With the above notation,

- (i) the ideal I of R generated by $\{[x, y]: x, y \in \overline{S}(R)\}$ is contained in $\overline{S}(R)$ and if $I = 0$ then $S(R) = \overline{S}(R)$ is a commutative subring of R , and
- (ii) if $S(R)$ is a commutative subring of R then, for every $s \in S(R)$, the ideal of R generated by $[s, R]$ is nilpotent of index at most 2.

Proof. (i) See [3, proof of Lemma 1.3].

(ii) See [7, p. 406].

Lemma 5 (see [4, Theorem 3.10]). With the above notation, suppose that R is noncommutative and semiprime and $S = S(R)$ is a field. Then

- (i) S is the centre of R .
- (ii) R is a division ring with $[R: S] = 4$ or R is isomorphic to the ring $M_2(S)$ of 2×2 matrices over S .

Lemma 6 (see [8, Theorem 1]). Suppose that K is a field of characteristic $\neq 2$ and $*$ is a K -involution of the ring $M_2(K)$. Then, in an appropriate K -basis, $*$ is one of the following types:

$$(\text{transpose involution}) \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

or

$$(\text{symplectic involution}) \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

or all a, b, c, d in K . In the former case, $S(M_2(K))$ is not a subring of $M_2(K)$.

Lemma 7. *With the above notation, suppose that $S = S(R)$ is a central subring of R . Suppose that M is an R -module and $\alpha \geq 0$ an ordinal such that*

- (i) $k_S(M) \not\leq \alpha$,
- (ii) $k_S(M/IM) < \alpha$, for every nonzero $*$ -ideal I of R , and
- (iii) $k_S(N) \leq \alpha$ or $k_S(M/N) < \alpha$, for every R -submodule N of M .

Then

- (a) $k_{(S+sR)}(M) \leq \alpha$ for every nonzero s in S , and
- (b) R is semiprime and no nonzero element of S is a zero-divisor in R .

Proof. (a) Let $0 \neq s \in S$. Suppose that $M = N_0 \supset N_1 \supset N_2 \supset \dots$ are $(S+sR)$ -submodules of M such that, for every $i \geq 1$, $k_{(S+sR)}(N_{i-1}/N_i) \not\leq \alpha$. Obviously, if $N = RN_1$ then $k_S(N) \not\leq \alpha$. Hence, by (iii), $k_S(M/N) < \alpha$. This implies that $k_S(sM/sN) < \alpha$. By (ii), $k_S(M/sN) < \alpha$. But $sN = sRN_1 \subseteq N_1$. Hence $k_{(S+sR)}(M/N_1) \leq k_S(M/N_1) < \alpha$, a contradiction.

(b) If $0 \neq s \in S$ and s is a zero-divisor in R then there exists a nonzero $*$ -ideal I of R such that $sI = 0$. By (ii), $k_S(M/IM) < \alpha$ and $k_S(M/sM) < \alpha$. But $k_S(sM) = k_S(sM/sIM) \leq k_S(M/IM) < \alpha$, and consequently, $k_S(M) < \alpha$, a contradiction.

Suppose now that J is a nonzero $*$ -ideal of R such that $J^2 = 0$. Obviously, $J \cap S = 0$, so that, for every $a \in J$, $a + a^* = 0$. Therefore, for any $r \in R$ and $a \in J$, $-ra = (ra)^* = a^*r^* = -ar^*$. Hence $aM = aRM = RaRM$ for any $a \in J$. Let $0 \neq a \in J$. Then, RaR is a nonzero $*$ -ideal and, by (ii), $k_S(M/JM) \leq k_S(M/aM) < \alpha$. Now $k_S(aM) = k_S(aM/aJM) \leq k_S(M/JM) < \alpha$. Thus $k_S(M) < \alpha$, a contradiction. It follows that R is semiprime.

Let R be a ring and M an R -module. For any nonempty subset X of R let $\text{Ann}_M(X) = \{m \in M : xm = 0 \text{ for all } x \in X\}$, the annihilator of X in M . The next result is the “dual” of Lemma 7.

Lemma 7*. *With the above notation, suppose that $S = S(R)$ is a central subring of R . Let M be an R -module and $\alpha \geq 0$ and ordinal such that*

- (i)* $k_S^*(M) \not\leq \alpha$,
- (ii)* $k_S^*(\text{Ann}_M(I)) < \alpha$, for every nonzero $*$ -ideal I of R , and
- (iii)* $k_S^*(N) < \alpha$ or $k_S^*(M/N) \leq \alpha$, for every R -submodule N of M .

Then

- (a)* $k_{(S+sR)}^*(M) \leq \alpha$, for every nonzero s in S , and
- (b)* R is semiprime and no nonzero element of S is a zero-divisor in R .

Proof. (a)* Let $0 \neq s \in S$. Suppose that $k_{(S+sR)}^*(M) \not\leq \alpha$. Then there exists a chain $0 = N_0 \subset N_1 \subset N_2 \subset \dots$ of $(S+sR)$ -submodules N_i ($i \geq 0$) of M such that, for every $i \geq 0$, $k_{(S+sR)}^*(N_{i+1}/N_i) \not\leq \alpha$. Put $N = RN_1$. Obviously, $sN \subseteq N_1$, so that $k_S^*(M/sN) \not\leq \alpha$. Hence $k_S^*(sN) < \alpha$, by (iii)*. However, $k_S^*(N_1) \not\leq \alpha$ gives $k_S^*(N) \not\leq \alpha$. But $sN \cong N/(\text{Ann}_N(Rs))$, hence $k_S^*(\text{Ann}_N(Rs)) \not\leq \alpha$, contradicting (ii)*.

(b)* Suppose that $0 \neq s \in S$ and s is a zero-divisor in R . Then there exists a nonzero $*$ -ideal I of R such that $sI = 0$. Consequently, $sM \subseteq \text{Ann}_M(I)$.

By (ii)*, $k_S^*(sM) < \alpha$. But $sM \cong M/(\text{Ann}_M(Rs))$, and again by (ii)*, $k_S^*(\text{Ann}_M(Rs)) < \alpha$. Thus $k_S^*(M) < \alpha$, contradicting (i)*.

Now suppose that J is a nonzero *-ideal of R such that $J^2 = 0$. Let $0 \neq a \in J$. As in the proof of Lemma 7, RaR is a nonzero *-ideal and $Ra = aR$, so that $\text{Ann}_M(a) = \text{Ann}_M(RaR)$. By (ii)*, $k_S^*(\text{Ann}_M(a)) < \alpha$. But $aM \cong M/(\text{Ann}_M(a))$, so, by (i)*, $k_S^*(aM) < \alpha$. Applying (ii)* again, we have $k_S^*(\text{Ann}_M(J)) < \alpha$. But $Ja = 0$ implies that $aM \subseteq \text{Ann}_M(J)$, and hence $k_S^*(aM) < \alpha$, a contradiction.

Now we are ready to prove our theorem.

Theorem. Suppose that R is a ring with involution $*$ such that $1/2 \in R$ and R is Noetherian with respect to two-sided *-ideals. Then $k_R(M) = k_{S(R)}^-(M)$ and $k_R^+(M) = k_{S(R)}^+(M)$, for every R -module M .

Proof. Applying Proposition 2 and Lemma 4(i), we can assume that $S(R) = S(R)$ is commutative. Since R is Noetherian with respect to two-sided *-ideals, it follows that the prime radical of R is nilpotent. Hence, applying Propositions 2 and 3(ii), we can assume that R is semiprime. Now Lemma 4(ii) implies that $S = S(R)$ is central in R .

Suppose that there exists an R -module M such that $k_R(M) = \alpha$ but $k_S(M) \neq \alpha$ (respectively, $k_R^+(M) = \alpha$ but $k_S^+(M) \neq \alpha$). Proposition 1 allows us to assume that, for every R -submodule N of M , $k_S(N) \leq \alpha$ or $k_S(M/N) < \alpha$ ($k_S^+(N) < \alpha$ or $k_S^+(M/N) \leq \alpha$). Let T be a two-sided *-ideal of R maximal such that $k_S(M/TM) \not\leq \alpha$ ($k_S^*(\text{Ann}_M(T)) \not\leq \alpha$). Suppose that there exists a two-sided *-ideal I of R , properly containing T , such that $k_S(M/IM) \not\leq \alpha$ ($k_S^*(\text{Ann}_M(I)) \not\leq \alpha$). Then $k_S(IM) \leq \alpha$ and $k_S(M/IM) = \alpha$, so that $k_S(M) = \alpha$, a contradiction (respectively, $k_S^+(M/\text{Ann}_M(I)) \leq \alpha$ and $k_S^*(\text{Ann}_M(I)) = \alpha$, so that $k_S^+(M) = \alpha$, a contradiction). Now replacing M by M/TM (respectively, M by $\text{Ann}_M(T)$) and R by R/T , if necessary, we can suppose without loss of generality, that in addition $k_S(M/IM) < \alpha$ ($k_S^*(\text{Ann}_M(I)) < \alpha$) for every nonzero two-sided *-ideal I of R .

Now suppose that R is a commutative ring. In this case, the involution $*$ is an automorphism of R . Hence $k_S(M) = \alpha$ (respectively, $k_S^+(M) = \alpha$), by Proposition 3, and this gives a contradiction. Thus R is not commutative. Applying Lemma 7 (Lemma 7*), we can form the ring of quotients A of R with respect to the nonzero elements of S . Clearly the involution $*$ can be extended uniquely to A and in this case $B = S(A)$ is the field of fractions of the domain S . Now Lemmas 5 and 6 allow us to assume that A is the ring of 2×2 matrices over B with symplectic involution or that A is a division ring with centre B and $|A:B| = 4$.

Suppose first that A is the ring of 2×2 matrices over B with the symplectic involution. Let $\{e_{ij} : i, j = 1, 2\}$ denote the set of matrix units of A . There exists $0 \neq s \in S$ such that $se_{ij} \in R$ ($i, j = 1, 2$). For any r in R , $r = \sum b_{ij}e_{ij}$, for some $b_{ij} \in B$. Let

$$u = s^2 b_{ij} e_{11} = s^2 e_{1i} r e_{j1} = (se_{1i}) r (se_{j1}) \in R.$$

But the involution on A is symplectic, and this gives in particular that

$$s^2 b_{ij} = s^2 b_{ij} (e_{11} + e_{22}) = u + u^* \in S.$$

Thus $s^3r = \sum (s^2b_{ij})(se_{ij}) \in \sum S(se_{ij})$. It follows that

$$s^3R \subseteq \sum S(se_{ij}) \subseteq R.$$

By Lemma 7, $k_{(S+s^3R)}(M) = \alpha$ (respectively, by Lemma 7*, $k_{(S+s^3R)}^*(M) = \alpha$) is that $k_U(M) = \alpha$ ($k_U^*(M) = \alpha$), where $U = S + \sum S(se_{ij})$.

Define $\sigma: U \rightarrow U$ by $\sigma(u) = (e_{11} - e_{22})u(e_{11} - e_{22})$ ($u \in U$). Obviously, σ is an automorphism of U and $\sigma^2 = 1$. Moreover,

$$V = U^\sigma = \{(s_1 + ss_2)e_{11} + (s_1 + ss_3)e_{22} : s_1, s_2, s_3 \in S\}.$$

By Proposition 3(i), $k_V(M) = \alpha$ ($k_V^*(M) = \alpha$). Now define $\rho: V \rightarrow V$ by $\rho(ve_{11} + we_{22}) = we_{11} + ve_{22}$, in the obvious notation. Clearly, ρ is an automorphism of V and $\rho^2 = 1$. Moreover $V^\rho = S$. By Proposition 3(i), $k_S(M) = \alpha$ (respectively, $k_S^*(M) = \alpha$).

Thus we can assume that A is a division ring with centre B and $|A:B| = 4$. Let $a \in A$ such that $B(a)$ is a maximal subfield of A . Clearly $B(a)$ is a splitting field for A . Moreover, $|B(a):B| = 2$, so there exists $b, c \in B$ such that $a^2 + ba + c = 0$. There exists a nonzero element t in S such that $tb, tc \in S$. Put $g(x) = x^2 + tbx + t^2c$. Clearly, $g(ta) = 0$. Further, the ring $R' = R[x]/(g(x))$ is a free R -module of rank 2. Hence the R -module

$$N = R' \otimes_R M \cong M \oplus M.$$

Thus $k_R(N) = \alpha$ (respectively, $k_R^*(N) = \alpha$). The involution $*$ induces an involution in the ring R' by

$$(r_0 + r_1x + (g(x))^* = r_0^* + r_1^*x + (g(x)) \quad (r_0, r_1 \in R),$$

and the set S' of symmetric elements of R' is precisely the ring $S[x]/(g(x))$. Localizing R' at the set of nonzero elements of S , we obtain a ring isomorphic to the ring of 2×2 matrices over $B(a)$ with symplectic involution, by Lemma 6. By the foregoing, $k_{S'}(N) = \alpha$ ($k_{S'}^*(N) = \alpha$). Now observe that the mapping $\tau: S' \rightarrow S'$, defined by

$$\tau(x + (g(x))) = -tb - x + (g(x)),$$

is an automorphism of S' with $\tau^2 = 1$ and $(S')^\tau \cong S$. Thus, by Proposition 3, $k_S(N) = \alpha$, and hence $k_S(M) = \alpha$ (respectively, $k_S^*(N) = \alpha$ and hence $k_S^*(M) = \alpha$), a contradiction. This completes the proof of the theorem.

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