PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 2, June 1994

MINIMAL SURFACES WITH CONSTANT KÄHLER ANGLE IN COMPLEX PROJECTIVE SPACES

XIAOHUAN MO

(Communicated by Peter Li)

ABSTRACT. Let $\psi: S^2 \to CP^n$ be an isometric minimal immersion of the Riemann sphere S^2 into CP^n with constant Kähler angle θ. In this paper, we prove that Bolton et al.'s conjecture holds if θ is not too close to $\frac{\pi}{2}$, that is, ψ is $\pm$ holomorphic or belongs to the Veronese sequence if $|\cos\theta| \geq \frac{1}{5}$.

1. INTRODUCTION

Let N be a Kähler manifold with the complex structure J and the standard Kähler metric $\langle\ ,\ \rangle$, let M be a Riemann surface; and let $\psi: M \to N$ be an isometric minimal immersion of M into N. Then the Kähler angle θ of ψ, which is an invariant of the immersion ψ related to J, is defined by $\cos(\theta) = \langle Je_1, e_2\rangle$, where $\{e_1, e_2\}$ is an orthonormal basis of M. The Kähler angle gives a measure of the failure of ψ to be a holomorphic map. Indeed ψ is holomorphic if and only if $\theta = 0$ on N, while ψ is antiholomorphic if and only if $\theta = \pi$ on M. In [2] Chern and Wolfson pointed out that the Kähler angle of ψ plays an important role in the study of minimal surfaces in N. From this point of view, we would like to know all isometric minimal immersions of the constants Kähler angle in N.

In this paper, we shall discuss this problem when N is a complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4, denoted by CP^n. Let $S^2(K)$ be a 2-dimensional sphere of constant Gaussian curvature K. Examples of minimal surfaces of constant Kähler angle in CP^n are given in [1] and [3]. For example, for each integer p with $0 \leq p \leq n$, there exist full isometric minimal immersions $\psi_{n,p}: S^2(K_{n,p}) \to CP^n$, where $K_{n,p} = 4\backslash[n + 2p(n-2)]$. Each $\psi_{n,p}$ possesses holomorphic rigidity; that is, two such immersions differ by a holomorphic isometry of CP^n.

Characterizing minimal surfaces of constant Kähler angle in CP^n, Bolton et al. [1] conjectured that, if the Kähler angle of an isometric minimal immersion $\psi: S^2 \to CP^n$ is constant, then the Gaussian curvature of ψ is also constant, when the immersion is neither holomorphic, antiholomorphic, nor totally real. They gave an affirmative answer to this conjective for $n \leq 4$. We would like

Received by the editors July 27, 1992 and, in revised form, September 28, 1992.
1991 *Mathematics Subject Classification.* Primary 53A10; Secondary 49Q05, 53C42, 53C55.
This work was supported by the Postdoctor Foundation in China.

to discuss this conjecture under some additional conditions. We prove the following

Main Theorem. *Let CP^n be an n-dimensional complex projective space of constant holomorphic sectional curvature 4 and let S^2 be a topological 2-sphere. Let $\psi: S^2 \to CP^n$ be a linearly full isometric minimal immersion with constant Kähler angle θ. Then ϕ is $\pm$ holomorphic or belongs to the Veronese sequence $\psi_{n,p}$ if $|\cos\theta| \geq \frac{1}{5}$.*

2. Conjecture of Kähler angle

Let $\psi: S^2 \to CP^n$ be a linear full isometric minimal immersion. Let $\psi_0, \dots, \psi_n$ be the harmonic sequence determined by ψ with $\psi = \psi_q$ for some $q = 0, \dots, n$. Let $L_0, \dots, L_n$ be its associated sequence of complex line subbundles of the trivial bundle $S^2 \times C^{n+1}$. If

$$\partial_p: T^{(1,0)}S^2 \otimes L_p \to L_{p+1}, \qquad p = 0, \dots, n-1,$$

are the associated bundle holomorphisms described on [1, p. 602] then the ramification index $r(\partial_p)$ of ∂_p is a nonnegative integer. Let γ_p denote the type $(1, 0)$ energy density of ψ_p. Then the Kähler angle of ψ_q, denoted by θ_q, satisfies

$$t_p = (\tan(\theta_q/2))^2 = \gamma_{p-1}/\gamma_p.$$

We set

$$\delta_q = \frac{1}{2\pi i}\int_{S^2} \gamma_q \, d\bar{z} \wedge dz > 0$$

for $q = 0, \dots, n-1$. Here z is a local complex coordinate on S^2.

Suppose t_q constant.

Lemma 1. *If $t_q \leq p/(p+1)$ for some $p \in \{1, \dots, n\}$, then $q \leq p-1$.*

Proof. By the hypothesis, $(p+1)\gamma_{q-1}/p \leq \gamma_q$. From [1, Proof of Lemma 9.4, p. 619]

$$\frac{1}{2\pi i}\left(\delta_q - \frac{p+1}{p}\delta_{q-1}\right)\int_{S^2} \gamma_{q-1} \, d\bar{z} \wedge dz \geq 0.$$

From this we can conclude that $p\delta_q - (p+1)\delta_{q-1} \geq 0$.

Suppose that $q \geq p \geq 1$. Then, from [1, (3,24)], it follows that

$$\begin{aligned}
-p\delta_q + (p+1)\delta_{q-1} &= (p+1)q + (n-q)(q-p) \\
&\quad + \frac{(n-q+1+p)}{n+1}\sum_{k=0}^{q-2}(k+1)r(\partial_k) \\
&\quad - \frac{p}{n+1}(n-q)qr(\partial_{q-1}) + \frac{q-p}{n+1}\sum_{k=q}^{n-1}(n-k)r(\partial_k) \\
&\quad + \frac{(p+1)q(n-q+1)}{n+1}r(\partial_{q-1}) \\
&> \frac{(p+1+n-q)q}{n+1}r(\partial_{q-1}) \geq 0.
\end{aligned}$$

Hence $p\delta_q - (p+1)\delta_{q-1} < 0$ which contradicts the estimate above. Hence $q \leq p-1$.

Lemma 2. *If* $t_q \geq (p+1)/p$ *for some* $p \in \{1, \ldots, n\}$, *then* $q \geq n - p + 1$.

Proof. The proof is similar to the proof of Lemma 1.

Lemma 3. $|\cos\theta| \geq \frac{1}{5}$ *if and only if either* $t_q \leq \frac{2}{3}$ *or* $t_q \geq \frac{3}{2}$.

Proof. Trig identities prove Lemma 3.

Theorem. *If* $|\cos\theta| \geq \frac{1}{5}$, *then* ψ *is part of the Veronese sequence, holomorphic, or antiholomorphic.*

Proof. By Lemma 3, $t_q \leq \frac{2}{3}$ or $t_q \geq \frac{3}{2}$, say the former holds. Then the hypothesis of Lemma 1 holds for $p = 2$, and therefore $q \leq 1$. If $q = 0$, then ψ is holomorphic. If $q = 1$, then $t_0 = 0$ and t_1 are both constant, so ψ is part of the Veronese sequence by [1, Remark 5.5, p. 612]. Similarly, if $t_q \geq \frac{3}{2}$, then $q \geq n - 2 + 1 = n - 1$, so either $q = n - 1$ or $q = n$. Again either ψ is antiholomorphic or ψ is part of the Veronese sequence by [1, Remark 5.5, p. 612].

References

1. John Bolton, Gary R. Jensen, Marco Rigoli, and Lyndon M. Woodward, *On conformal minimal immersions of* S^2 *into* CP^n, Math. Ann. **279** (1988), 599–620.
2. S. S. Chern and J. G. Wolfson, *Minimal surfaces by moving frame*, Amer. J. Math. **105** (1983), 59–83.
3. S. Bando and Y. Ohnita, *Minimal 2-spheres with constant curvature in* $P_n(C)$, J. Math. Soc. Japan **39** (1987), 477–487.

Institute of Mathematics, Fudan University, Shanghai 200433, China
Current address: Department of Mathematics, Peking University, Beijing, 100871, China