COMBINATORICS AND TOPOLOGY OF LINE ARRANGEMENTS IN THE COMPLEX PROJECTIVE PLANE

ENRIQUE ARTAL-BARTOLO

(Communicated by Ronald Stern)

ABSTRACT. We use some results about Betti numbers of coverings of complements of plane projective curves to discuss the problem of how combinatorics determine the topology of line arrangement, finding a counterexample to a conjecture of Orlik.

1. INTRODUCTION

In [E], a formula for Betti numbers of the Milnor fiber of the cone over a projective plane curve is given involving the dual of some differentials which arise in a spectral sequence. We have presented this formula in a more precise form in [A], showing an explicit relation with the infinitely near points. In this paper we are going to apply these results to the theory of line arrangements in the complex projective plane $\mathbb{P}^2$.

Let $f \in \mathbb{C}[X, Y, Z]$ be a reduced homogeneous polynomial of degree N and $C := \{[x : y : z] \in \mathbb{P}^2 \mid f(x, y, z) = 0\}$ the curve defined by f ; let r be the number of global irreducible components. Let $E := \mathbb{P}^2 \setminus C$, be the complement of the curve and let $X := \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = 1\}$; X is the Milnor fiber of the singularity defined by f , namely, the cone over C . The projection $\tau: X \rightarrow E$, such that $\tau(x, y, z) := [x : y : z]$, is an N -fold cyclic covering. Its automorphism group is μ_N , the group of the N th roots of unity in $\mathbb{C}$; an element $\zeta \in \mu_N$ acts by $\zeta \cdot (x, y, z) = (\zeta x, \zeta y, \zeta z)$.

We are going to recall the formula for $b_1(X)$ in terms of the position of singularities of C and some annihilation theorems from [Z1, E].

In the last section we establish another proof of the nonexistence of curves of degree 7 (resp. 8) with 11 (resp. 16) cusps; these results come from [Z2]. We end with some applications to the theory of hyperplane arrangements. We study the next conjecture, communicated to the author by P. Orlik.

1.1. Conjecture. Let C be a reduced projective plane curve of degree N whose irreducible components are lines. Let $\zeta: X \rightarrow X$ be a generator of the automorphism group of the covering $\tau: X \rightarrow E$. Then $\zeta^*: H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathbb{C})$ is the identity.

Received by the editors September 28, 1992.

1991 *Mathematics Subject Classification*. Primary 14N10, 14F45; Secondary 14F25, 32S55.

Key words and phrases. Singularity, Milnor fiber, combinatorics, Betti numbers, arrangements.

1.2. *Remark.* The quotient of $H^*(X, \mathbb{C})$ by the action of ζ^* is identified with the complex cohomology of the quotient of X by the action of ζ . This quotient is the base E of the covering. Then, this conjecture is equivalent to

$$b_1(X) = b_1(E), \quad b_2(X) = b_2(E) + (N-1)\chi(E).$$

We find counterexamples to this conjecture.

2. THE FORMULA FOR $b_1(X)$

Let us consider $C \in \mathbb{P}^2$. We consider $\text{Sing}(C)$ the set of singular points of C .

Let $\sigma: Y \rightarrow \mathbb{P}^2$ the minimal embedded resolution of the singular points of C . We set

$$D := \sigma^*C = C_1 + \cdots + C_r + \sum_{P \in \text{Sing}(C)} \sum_{j=1}^{l_P} m_j^P E_j^P,$$

the total transform of C where $C_1, \dots, C_r$ are the irreducible components of the strict transform of C and $E_1^P, \dots, E_{l_P}^P$ are the irreducible components of $E_P := \sigma^{-1}(P)$, which is the exceptional locus of σ at P , for each $P \in \text{Sing}(C)$. No smooth point is blown up and then, the multiplicities m_j^P are at least two.

We consider $\mathcal{O}(1)$, the rank one, locally free sheaf on $\mathbb{P}^2$ whose Chern class is 1; we consider $\mathcal{O}(1) = \mathcal{O}(H)$ where H is a line in $\mathbb{P}^2$ in general position with respect to C . We identify H with $\sigma^*(H)$. $\mathcal{L} := \sigma^*\mathcal{O}(1)$ is a sheaf on Y and $\mathcal{L} = \mathcal{O}(H)$.

2.1. **Definition.** The k th sheaf associated to C , $k = 0, \dots, N-1$, is the sheaf on Y defined by

$$\mathcal{L}^{(k)} := \mathcal{L}^k \otimes \mathcal{O} \left(- \sum_{P,j} \left[\frac{m_j^P k}{N} \right] E_j^P \right).$$

The next result of [E] uses the Mixed Hodge Structure on the cohomology of X .

2.2. **Proposition.** Let $H^1(X; \mathbb{C}) := \bigoplus_{k=0}^{N-1} H^k$ be the decomposition of $H^1(X; \mathbb{C})$ in eigenspaces for the action of the monodromy of the covering τ , where the eigenvalue of H^k is $\exp(2i\pi k/N)$. Then, $\dim H^0 = r-1$ and H^k is naturally isomorphic to $H^1(Y, \mathcal{L}^{(k)}) \oplus \overline{H^1(Y, \mathcal{L}^{(N-k)})}$, for $k = 1, \dots, N-1$.

2.3. Using the adjunction formula [BPV, Chapter I], the canonical sheaf of Y may be presented as

$$\mathcal{K}_Y = \mathcal{O}(K_Y) = \mathcal{O} \left(-3H + \sum_P \sum_j \kappa_j^P E_j^P \right)$$

where κ_j^P are the only solution of

$$E_j^P \cdot \left(\sum_l \lambda_l^P E_l^P \right) = -2 - E_j^P \cdot E_j^P.$$

2.4. Definition. Let us fix $P \in \text{Sing}(\mathbf{C})$ and $k \in \{1, \dots, N-1\}$. We define as $\mathcal{I}_{P,k,N}$ the ideal of $\mathcal{O}_{\mathbb{P}^2,P}$ whose elements are the germs h of holomorphic functions at P which can be lifted by σ as sections of $\mathcal{O}(\sum(\kappa_j^P - [m_j^P k/N]E_j^P))$ in the neighbourhood of the exceptional locus of σ over P .

We define a homomorphism $\sigma_k: H^0(\mathbb{P}^2, \mathcal{O}(k-3)) \rightarrow \sum_{P \in \text{Sing}(\mathbf{C})} \mathcal{O}_{\mathbb{P}^2,P} / \mathcal{I}_{P,k,N}$ as follows: Let X, Y, Z be homogenous coordinates of $\mathbb{P}^2$ such that $Z=0$ is the equation for L . Then, the domain of σ_k is identified with the set of polynomials in X, Y of degree not bigger than $k-3$. The image of a polynomial g is the formal sum of the germs of g at P , $P \in \text{Sing}(\mathbf{C})$, regarded as elements of the quotient rings.

The kernel of σ_k is the set of equations of plane projective curves of degree $k-3$ which pass through the points of $\text{Sing}(\mathbf{C})$ following the instructions given by the ideals $\mathcal{I}_{P,k,N}$. These instructions impose conditions on the *Enriques' infinitely near points* of the curves. The next formula is proved in [A].

2.5. Theorem. *The dimension of H^k is $\sum_{P \in \text{Sing}(\mathbf{C})} d_k^P - ((k-1)(k-2)/2 + \dim \ker \sigma_k)$ where*

$$d_k^P = \frac{1}{2} \left(\sum_{a \in S(P)} \left[\frac{m_a k}{d} \right] E_a \right) \cdot \left(\sum_{a \in S(P)} \left(\kappa_a - \left[\frac{m_a k}{d} \right] \right) E_a \right).$$

In fact, $\dim H^k \leq \sum_{P \in \text{Sing}(\mathbf{C})} d_k^P$.

In the next section we will use this formula with two annihilation theorems.

2.6. Theorem [Z2]. *If $\mathbf{C}$ is irreducible and N is a prime power, then $H^1(X, \mathbb{C}) = 0$.*

2.7. Proposition [E]. *Let $k \in \{1, \dots, N-1\}$; let us suppose that $m_j^P k/N$ is never an integer number. Then, $H^k = 0$.*

This last result is more general than Esnault's result but the proof is essentially the same.

3. EXAMPLES AND HYPERPLANE ARRANGEMENTS

3.1. Example. There is no curve of degree 7 and 11 cusps in the singular locus.

For a cusp, the multiplicities in the exceptional divisor are 2, 3, 6. Because of Proposition 2.7, $b_1(X) = r-1$. Nevertheless, if we compute the coefficients a_j^P of the canonical divisor, we get 1, 2, 4, for each P . We find that $d_6^P = 1$ for each P . As $\dim H^0(\mathbb{P}^2, \mathcal{O}(3)) = 10$, we have that $b_1(X) > r-1$, and then, a contradiction.

3.2. Example. There is no curve of degree 8 and 16 cusps in the singular locus.

As in example 3.5 we see that $d_7 = 16$. Because $\dim H^0(\mathbb{P}^2, \mathcal{O}(4)) = 15$, we have that $b_1(X) > r-1$. We also have $d_k = 0$ if $k < 7$; as 7 is prime to 2, 3, and 6, it implies that $b_1(X) = r-1$ because of Proposition 2.7, a contradiction.

From now on, we are only interested in curves $\mathbf{C}$, having only ordinary singular points. That is to say, if $P \in \text{Sing}(\mathbf{C})$, $\mathbf{C}$ has m_P pairwise transversal smooth local components at P . The surface Y is obtained by blowing up the points $P \in \text{Sing}(\mathbf{C})$. Then, the exceptional divisor of σ has $\#\text{Sing}(\mathbf{C})$ pairwise

disjoint irreducible components, denoted by E_P ; they are projective lines with self-intersection -1 and multiplicity m_P . In this case, the divisors D and K_Y of last paragraph are

$$D = C_1 + \cdots + C_r + \sum_{P \in \text{Sing}(C)} m_P E_P, \quad K_Y = -3L + \sum_{P \in \text{Sing}(C)} E_P.$$

For k and P , the divisor involved in the definition of $\mathcal{I}_{P,k,N}$ is $(1 - [m_P k/N])E_P$.

Using local coordinates at $P \in \text{Sing}(C)$ we see $\mathcal{I}_{P,k,N} = \mathcal{M}_P^{[m_P k/N]-1}$, where $\mathcal{M}_P$ is the maximal ideal of $\mathcal{O}_{\mathbb{P}^2, P}$.

We remark that

$$\dim \mathcal{O}_{\mathbb{P}^2, P} / \mathcal{I}_{P,k,N} = \sum \frac{[m_P k/N]([m_P k/N] - 1)}{2} = d_k^P.$$

Therefore,

$$b_1(X) = r - 1 + 2 \sum_{k=1}^{N-1} \dim \text{coker } \sigma_k,$$

where

$$\sigma_k: H^0(\mathbb{P}^2, \mathcal{O}(k-3)) \rightarrow \sum_{P \in \text{Sing}(C)} \frac{\mathcal{O}_{\mathbb{P}^2, P}}{\mathcal{M}_P^{[m_P k/N]-1}}.$$

3.3. Remark. The kernel of σ_k is the set of equations of the curves of degree $k-3$ passing through $P \in \text{Sing}(C)$ with multiplicity at least m_P .

We are going to apply these results to line arrangements. We can find most definitions and results in [O]. A line arrangement is a curve C of degree N which decomposes into N different lines. We want to know which topological invariants of the couple $(\mathbb{P}^2, C)$ only depend on the combinatorics of the arrangement, i.e., how the N lines do intersect.

The cohomology of the complement E of C is well known; $b_1(E) = N - 1$ is a consequence of the following general result:

3.4. Lemma. Let $C_1, \dots, C_r$ irreducible curves of degrees $d_1, \dots, d_r$. Then

$$H^1\left(\mathbb{P}^2 \setminus \bigcup_{i=1}^r C_i; \mathbb{Z}\right) = |\mu_1, \dots, \mu_r: d_1 \cdot \mu_1 + \cdots + d_r \cdot \mu_r = 0|.$$

In this case it is easy to calculate $b_2(E)$ using the facts that the Milnor number of an ordinary singularity of multiplicity m is $\mu = (m-1)^2$ and $\chi(E) = N^2 - 3N + 3 - \sum_{P \in \text{Sing}(C)} \mu_P$.

The results of the last paragraph give some new information about X ; this space is not well known. These results allow us to give some counterexamples to the conjecture stated in the introduction.

3.5. Example. Let C_N be an arrangement of N lines passing through the point $[0:0:1]$. We have

$$\sigma_k: H^0(\mathbb{P}^2, \mathcal{O}(k-3)) \rightarrow \mathbb{C}\{x, y\}/(x, y)^{k-1}.$$

This map is one-to-one, as a curve of degree $k-3$ cannot have points of multiplicity $k-1$. Then

$$\dim \text{coker } \sigma_k = \frac{k(k-1)}{2} - \frac{(k-1)(k-2)}{2} = k-1.$$

FIGURE 3.1

We have just seen that the monodromy does not act trivially on $H^1(X, \mathbb{C})$, and

$$b_1(X) = N - 1 + 2 \sum_{k=1}^{N-1} (k - 1) = (N - 1)^2.$$

3.6. Example. Let us consider the arrangement of lines (Figure 3.1) passing through points $P_1, \dots, P_4$, such that no three of them are in a line. There are four triple points. We are only interested in σ_4 :

$$\sigma_4: H^0(\mathbb{P}^2, \mathcal{O}(1)) \rightarrow \sum_{i=1}^4 \frac{\mathcal{O}_{P_i}}{\mathcal{M}_{P_i}}.$$

The kernel of this map is zero: $\dim \operatorname{coker} \sigma_4 = 1$, and therefore

$$b_1(X) = 6 - 1 + 2 \cdot 1 = 7.$$

One open problem in this theory is to know whether the topology of the Milnor fiber X depends only on the combinatorics. It is not known for the Betti numbers of X , though the answer is yes for the Betti numbers of the complement E ; it is also known that the ring $H^*(E; \mathbb{C})$ is a combinatorial invariant (see [O]).

We are going to consider two examples of arrangements with *not very different* combinatorics; the Betti numbers of E are the same, but it is not the case for the Betti numbers of X .

Both arrangements have nine lines, nine double points, nine triple points, and each line contains exactly three triple points. In both cases, $b_1(E) = 8$. On the other hand, $b_1(X) = 8 + 2 \dim \operatorname{coker} \sigma_6$, with

$$\sigma_6: H^0(\mathbb{P}^2, \mathcal{O}(3)) \rightarrow \sum_{i=1}^9 \frac{\mathcal{O}_{P_i}}{\mathcal{M}_{P_i}}.$$

We must find the cubics passing through the nine triple points.

3.7. Example. There exists an arrangement as above such that $b_1(X) = 10$.

Proof. There are three cubics Q_1, Q_2, Q_3 where each Q_i decomposes in three lines in general position. The nine lines are pairwise different and the three cubics have exactly nine common points. The arrangement $Q_1 \cup Q_2 \cup Q_3$ is the good one: Q_1 and Q_2 generate the set of cubics passing through the nine triple points. Then, $\dim \operatorname{coker} \sigma_6 = 1$ and $b_1(X) = 8 + 2 = 10$.

Such an arrangement exists:

$$\begin{aligned} Q_1 &= XZ(abX + (a - ab + 1)Y + bZ), & a(a+1) &\neq 0, \\ Q_2 &= Y(X - Y - Z)(aX + Y + bZ), & b(b-1) &\neq 0, \\ Q_3 &= (X - Y)(aX + Y + Z)(Y + bZ), & a - ab + 1 &\neq 0, \\ & & a + b &\neq 0, \\ & & ab + b - 1 &\neq 0. \end{aligned}$$

We have the equation of the three cubics. $\square$

3.8. Example. There exists an arrangement as above such that $b_1(X) = 8$.

Proof. We give the equations of an arrangement whose triple points determine one cubic; then, $\dim \operatorname{coker} \sigma_6 = 0$ and $b_1(X) = 8$. The existence of only one cubic guarantees that the combinatorics of this arrangement and that of the arrangement in Example 3.7 are different. The example is

$$\begin{aligned} Q &= X(X + Y)(cX + (c + 1)Y + bZ) & abc &\neq 0, \\ & Y(X + aZ)(c(a + b)X + a(c + 1)Y + ab(c + 1)Z) & c + 1 &\neq 0, \\ & Z(Y + bZ)(cX + (c + 1)Y + acZ), & ac - b &\neq 0, \\ & & a + b &\neq 0, \\ & & c(a - bc) - b(c + 1) &= 0. \end{aligned}$$

These equations give the end of the proof. $\square$

REFERENCES

- [A] E. Artal, *Les couples de Zariski*, J. Algebraic Geom. (to appear).
- [BPV] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Springer-Verlag, Berlin, Heidelberg, and New York, 1984.
- [E] H. Esnault, *Fibre de Milnor d'un cône sur une courbe algébrique plane*, Invent. Math. **68** (1982), 477–496.
- [O] P. Orlik, *Introduction to arrangements*, Proc. Sympos. Pure Math., vol. 72, Amer. Math. Soc., Providence, RI, 1989.
- [Z1] O. Zariski, *On the linear connection index of the algebraic surfaces*, Proc. Nat. Acad. Sci. U.S.A. **15** (1929), 494–501.
- [Z2] ———, *On the irregularity of cyclic multiple planes*, Ann. of Math. (2) **32** (1931), 485–511.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE ZARAGOZA, E-50009 ZARAGOZA, SPAIN
 Current address: Departamento de Geometría y Topología, Universidad Complutense, E-28040 Madrid, Spain
 E-mail address: artal@mat.ucm.es