ON THE SIZE OF FINITE SIDON SEQUENCES

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ABSTRACT. Let $h \geq 2$ be an integer. A set of positive integers B is called a B_h -sequence, or a Sidon sequence of order h , if all sums $a_1 + a_2 + \cdots + a_h$, where $a_i \in B$ ($i = 1, 2, \dots, h$), are distinct up to rearrangements of the summands. Let $F_h(n)$ be the size of the maximum B_h -sequence contained in $\{1, 2, \dots, n\}$. We prove that

$$F_{2r-1}(n) \leq ((r!)^2 n)^{1/(2r-1)} + O(n^{1/(4r-2)}).$$

Let $h \geq 2$ be an integer. A set of positive integers B is called a B_h -sequence if all sums $a_1 + a_2 + \cdots + a_h$, where $a_i \in B$ ($i = 1, 2, \dots, h$), are distinct up to rearrangements of the summands.

A B_h -sequence is also called a Sidon sequence of order h [6]. We say that B is a B_h -sequence for $\mathbb{Z}/(n)$ if B is a finite B_h -sequence and all sums are distinct modulo n .

Let $F_h(n)$ denote the size of maximum B_h -sequences contained in the set of integers $\{1, 2, \dots, n\}$ and $f_h(n)$ the size of maximum B_h -sequence for $\mathbb{Z}/(n)$. Then it follows from a simple combinatorial argument that

$$F_h(n) \leq (h \cdot h!)^{1/h} n^{1/h} \quad \text{and} \quad f_h(n) \leq (h!)^{1/h} n^{1/h}.$$

Erdős and Turan [2] proved that $F_2(n) < \sqrt{n} + O(n^{1/4})$. On the other hand, Bose and Chowla [1] showed that, for every $h \geq 2$, there exists a B_h -sequence B for $\mathbb{Z}/(m^h - 1)$ with $|B| = m$, where m is a prime power. This implies that

$$F_h(n) \geq (1 + o(1))n^{1/h}.$$

Therefore, $F_2(n) = (1 + o(1))\sqrt{n}$.

Erdős conjectured that $F_2(n) = \sqrt{n} + O(1)$. For $h = 3$, Lee [4] obtained that

$$F_3(n) \leq \left(\left(1 - \frac{1}{6 \log_2^2 n} \right) 4n \right)^{1/3}.$$

For $h = 4$, Lindström [5] proved that

$$F_4(n) \leq (8n)^{1/4} + O(n^{1/8}).$$

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When $h = 2r$ ($r \geq 1$), Jia [3] showed that

$$F_{2r}(n) \leq ((r!)^2 rn)^{1/2r} + O(n^{1/4r})$$

and

$$f_{2r}(n) \leq ((r!)^2 n)^{1/2r} + O(n^{1/4r}).$$

In this paper, we shall obtain a similar upper bound for $F_{2r-1}(n)$ and $f_h(n)$ as well.

Theorem 1. For all $r \geq 1$,

$$F_{2r-1}(n) \leq ((r!)^2 n)^{1/(2r-1)} + O(n^{1/(4r-2)}).$$

Theorem 2. For all $r \geq 1$,

$$f_{2r}(n) \leq ((r!)^2 n)^{1/2r} + O(1)$$

and

$$f_{2r-1}(n) \leq (r!(r-1)!n)^{1/(2r-1)} + O(1).$$

First, some notation. Let B be a B_{2r-1} -sequence. Let $A = rB$ where rB denotes the set of all sums of r not necessarily distinct elements in B . We have

$$t = |A| = \binom{k+r-1}{r} \geq \frac{k^r}{r!}.$$

Let

$$V = \{(a, b) | a, b \in rB\} = V_0 \cup V_1,$$

where $V_1 = V \setminus V_0$ and V_0 consists of all elements (a, b) such that $a = \sum_{i=1}^r a_i$ and $b = \sum_{i=1}^r b_i$ with $a_i, b_j \in B$ and $a_j \neq b_j$ for all $1 \leq i, j \leq r$.

Lemma. For any integer d , there are at most k/r elements (a, b) in V_0 such that $a - b = d$.

Proof. Let (a_i, b_i) be elements in V_0 , $i = 1, 2, \dots, s$, such that $a_i - b_i = d$ for all $1 \leq i \leq s$. Suffice to show that, if $s > k/r$, at least two of the (a_i, b_i) 's are the same.

Now assume $s > k/r$. Write $a_i = \sum_{j=1}^r a_{ij}$ and $b_i = \sum_{j=1}^r b_{ij}$ where $a_{ij}, b_{ij} \in B$. Since $|B| = k$ and $sr > k$, there are at least two distinct pairs (i, j) and (i', j') ($1 \leq i, i' \leq s$ and $1 \leq j, j' \leq r$) such that $a_{ij} = a_{i'j'}$. By the definition of V_0 , $i \neq i'$. But then we have $a_i - a_{ij} - b_i = a_{i'} - a_{i'j'} - b_{i'}$. Hence $a_i - a_{ij} + b_{i'} = a_{i'} - a_{i'j'} + b_i$. As B is a B_{2r-1} -sequence and a_{ij} 's and b_{ij} 's are all mutually distinct, we have $\{a_{i1}, a_{i2}, \dots, a_{ir}\} = \{a_{i'1}, a_{i'2}, \dots, a_{i'r}\}$ and $\{b_{i1}, b_{i2}, \dots, b_{ir}\} = \{b_{i'1}, b_{i'2}, \dots, b_{i'r}\}$. Therefore, $(a_i, b_i) = (a_{i'}, b_{i'})$. This completes the proof of the Lemma.

Proof of Theorem 1. Let $u = \lfloor n^{(4r-3)/(4r-2)} \rfloor$ and $I_m = [-u + m, -1 + m]$, $m = 1, 2, \dots, rn + u$, $C_m = I_m \cap B$, and $c_m = |C_m|$. Then

$$(1) \quad (tu)^2 = \left(\sum_{m=1}^{rn+u} c_m \right)^2 \leq (rn + u) \sum_{m=1}^{rn+u} c_m^2.$$

Note c_m^2 is the number of elements $(a, b) \in V$ such that $a, b \in C_m$. Hence $-u < a - b < u$.

For any integer d , $-u < d < u$, by the Lemma, there are at most k/r elements (a, b) in V_0 such that $a - b = d$. And each such pair is counted $u - |d|$ times toward the sum $\sum_{m=1}^{rn+u} c_m^2$. Hence, as $|V_1| \leq O(k^{2r-1})$ and $k \leq O(n^{1/(2r-1)})$,

$$\begin{aligned} (tu)^2 &\leq (rn + u) \left(\sum_{-u < d < u} \frac{k}{r} (u - |d|) + O(k^{2r-1}) \right) \\ &= (rn + u) \left(\frac{k}{r} u^2 + O(k^{2r-1}) \right). \end{aligned}$$

So,

$$t^2 \leq nk(1 + O(n^{-1/(4r-2)})).$$

Hence,

$$k \leq ((r!)^2 n(1 + O(n^{-1/(4r-2)})))^{1/(2r-1)} \leq ((r!)^2 n)^{1/(2r-1)} + O(n^{1/(4r-2)}).$$

This proves Theorem 1. $\square$

Proof of Theorem 2. In the case $h = 2r - 1$, we use the same settings A and V . Since, for any $d \in Z/(n)$, there exist at most k/r elements $(a, b) \in V_0$ such that $a - b = d$, we have

$$t^2 = |V| = |V_0| + |V_1| \leq \frac{k}{r}n + O(k^{2r-1}).$$

Hence,

$$\frac{k^{2r}}{(r!)^2} \leq \frac{k}{r}n + O(k^{2r-1}) = \frac{k}{r}n(1 + O(n^{-1/(2r-1)})),$$

and

$$\begin{aligned} k &\leq (r!(r-1)!n)^{1/(2r-1)}(1 + O(n^{-1/(2r-1)}))^{1/(2r-1)} \\ &= (r!(r-1)!n)^{1/(2r-1)} + O(1). \end{aligned}$$

This shows that $f_{2r-1}(n) \leq (r!(r-1)!n)^{1/(2r-1)} + O(1)$. Similarly, we have $f_{2r}(n) \leq ((r!)^2 n)^{1/(2r)} + O(1)$, which completes the proof of Theorem 2. $\square$

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