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EXOTIC SMOOTHINGS OF HYPERBOLIC MANIFOLDS WHICH DO NOT SUPPORT PINCHED NEGATIVE CURVATURE

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ABSTRACT. Constructed in this note are examples of topological manifolds M supporting at least two distinct smooth structures $\mathscr{M}_1$ and $\mathscr{M}_2$, where $\mathscr{M}_1$ is a complete, finite volume, real hyperbolic manifold, while $\mathscr{M}_2$ cannot support a complete, finite volume, pinched negatively curved Riemannian metric.

A Riemannian manifold M is said to be *pinched negatively curved* provided there exists a real number r with $r < -1$ such that each sectional curvature K of M satisfies the inequality $r \leq K \leq \frac{1}{r}$.

A given topological manifold M can often support many distinct smooth structures. We construct in this note examples of topological manifolds M supporting at least two distinct smooth structures $\mathscr{M}_1$ and $\mathscr{M}_2$, where $\mathscr{M}_1$ is a complete, finite volume, real hyperbolic manifold, while $\mathscr{M}_2$ *cannot* support a complete, finite volume, pinched negatively curved Riemannian metric. (Recall a real hyperbolic manifold is a Riemannian manifold, all of whose sectional curvatures are equal to -1.) This note supplements the results of papers [7] and [8] where the opposite phenomenon was studied. Namely, in these earlier papers examples of distinct smoothings $\mathscr{M}_1$ and $\mathscr{M}_2$ were constructed, where $\mathscr{M}_1$ is as above but $\mathscr{M}_2$ also supports a complete, finite volume, pinched negatively curved Riemannian metric. Compact examples were constructed in [7] and noncompact examples in [8].

Let M^m be a connected smooth manifold (with $m > 5$) and $f\colon \mathbb{R}\times\mathbb{D}^{m-1} \to M^m$ a smooth embedding which is also a proper map (i.e., $f^{-1}(K)$ is compact whenever K is a compact subset of M). We call f a *proper tube*. Let $\phi\colon S^{m-2} \to S^{m-2}$ be an orientation-preserving diffeomorphism of the sphere S^{m-2} and identify S^{m-2} with $\partial\mathbb{D}^{m-1}$. Form a new smooth manifold $M^m_{f,\phi}$ as the quotient space of the disjoint union

$$\mathbb{R}\times\mathbb{D}^{m-1} \coprod M^m - f(\mathbb{R}\times \operatorname{Int}\mathbb{D}^{m-1}),$$

where each point $(t, x) \in \mathbb{R}\times S^{m-2}$ is identified to the corresponding point $f(t, \phi(x)) \in M^m - f(\mathbb{R}\times\operatorname{Int}\mathbb{D}^{m-1})$. Recall that the isotopy class $[\phi]$ of ϕ is identified with an element in Θ_{m-1}—the group of oriented diffeomorphism

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classes of oriented homotopy $m-1$ spheres. Kervaire and Milnor [12] and Browder [1] showed that Θ_{m-1} is nontrivial for every even integer m greater than 4 and not of the form 2^n-2, $n\in\mathbb{Z}$.

Let $f\colon \mathbb{R}\times\mathbb{D}^{m-1}\to M^m$ be a proper tube. We say that it *connects different ends* of M^m if there exist a compact subset K in M^m and a positive real number r such that $f([r,+\infty)\times\mathbb{D}^{m-1})$ and $f((-\infty,-r]\times\mathbb{D}^{m-1})$ lie in different components of M^m-K.

We can now state the main result of this note.

Theorem 1. *Let M^m $(m>5)$ be a parallelizable complete real hyperbolic manifold with finite volume and $f\colon \mathbb{R}\times\mathbb{D}^{m-1}\to M^m$ a proper tube which connects different ends of M^m. If the diffeomorphism $\phi\colon S^{m-2}\to S^{m-2}$ represents a nontrivial element of Θ_{m-1}, then the smooth manifold $M^m_{f,\phi}$ is*

(1) *homeomorphic to M^m and*
(2) *does not support a complete pinched negatively curved Riemannian metric with finite volume.*

Corollary 2. *Let $m>5$ be any integer such that the group Θ_{m-1} is nontrivial (e.g., m can be any even integer greater than 4 and not of the form 2^n-2; in particular, m could be 8). Then there exists a connected smooth manifold N^m such that*

(1) *N^m is homeomorphic to a complete real hyperbolic manifold with finite volume, and*
(2) *N^n does not support a complete pinched negatively curved Riemannian metric with finite volume.*

Proof of Corollary 2. Let $\mathbb{Z}$ denote the additive group of integers. The argument in [8, §4, Proof of Theorem 0.1] is easily modified to yield a connected, complete, noncompact, parallelizable, real hyperbolic manifold M^m and an epimorphism $\lambda\colon \pi_1M^m\to\mathbb{Z}$ such that $\lambda(\beta)$ is divisible by 2 for every cuspidal element β in π_1M^m. (An element in π_1M^m is *cuspidal* if it is freely homotopic to an element in the fundamental group of a cusp of M^m.) Let $\mathcal{N}^m$ be the finite sheeted covering space of M^m corresponding to the subgroup $\lambda^{-1}(2\mathbb{Z})$ of π_1M^m. Then $\mathcal{N}^m$ has twice as many cusps as M^m. Consequently, $\mathcal{N}^m$ is connected and has at least two distinct cusps. Therefore, we can construct a proper tube $f\colon\mathbb{R}\times\mathbb{D}^{m-1}\to\mathcal{N}^m$ connecting different ends of $\mathcal{N}^m$. Let N^m be $\mathcal{N}^m_{f,\phi}$ where ϕ represents a nontrivial element of Θ_{m-1}. Now applying Theorem 1, in whose statement M^m is replaced by $\mathcal{N}^m$, we see that N^m satisfies the conclusions of Corollary 2.

The proof of Theorem 1 requires some preliminary results. The first is the following generalization of Bieberbach's rigidity theorem due to Lee and Raymond [15].

Lemma 3. *Let $f\colon M_1\to M_2$ be a homotopy equivalence between two closed infranilmanifolds M_1 and M_2. Then f is homotopic to a diffeomorphism.*

Remarks. Recall that an infranilmanifold is a double coset space $\Gamma\backslash F\ltimes N/F$, where N is a simply connected nilpotent Lie group, F is a finite group, and Γ is a torsionfree discrete subgroup of the semidirect product $F\ltimes N$. (It can also be assumed that $F\to\operatorname{Aut}N$ is a faithful representation.)

Lemma 4. *Let N^n $(n > 4)$ be a closed infranilmanifold which is also a π-manifold. Let Σ^n be a homotopy sphere which is not diffeomorphic to S^n. Then the connected sum $N^n \# \Sigma^n$ is not diffeomorphic to any infranilmanifold.*

Proof. Because of Lemma 3, it is sufficient to show that $N^n \# \Sigma^n$ is not diffeomorphic to N^n. Recall from [13, pp. 25 and 194] that the concordance classes of smooth structures on (the topological manifold) N^n can be put in bijective correspondence with the homotopy classes of maps from N^n to Top/0 denoted $[N^n, \mathrm{Top}/0]$ with the infranilmanifold structure corresponding to the constant map. Let $\rho: N^n \to S^n$ be a degree one map. It induces a map

$$\rho^*: \Theta_n = [S^n, \mathrm{Top}/0] \to [N^n, \mathrm{Top}/0],$$

and it can be shown that $\rho^*(\Sigma^n)$ is the concordance class of $N^n \# \Sigma^n$. Since N^n is a π-manifold, the argument proving [7, Claim 2.4] applies to show that ρ^* is monic. Hence $N^n \# \Sigma^n$ is not concordant to N^n. Then a slight modification of the argument given to prove [7, Addendum 2.3] shows that $N^n \# \Sigma^n$ is not diffeomorphic to N^n. The modification consists of using Lemma 3 in place of Mostow's rigidity theorem [14] and [5, Theorem 5.1] in place of [6, Corollary 10.6]. This completes the proof of Lemma 4.

Our final preliminary result strings together facts contained in [2] and [16].

Lemma 5. *Let M^m be a connected complete pinched negatively curved Riemannian manifold with finite volume. Then there exists a compact smooth manifold $\overline{M}^m$ such that*

(1) *the interior of $\overline{M}^m$ is diffeomorphic to M^m and*
(2) *the boundary of $\overline{M}^m$ is the disjoint union of a finite number of infranilmanifolds.*

Proof of Lemma 5. Almost flat Riemannian manifolds are defined and investigated by Gromov in [10]. Buser and Karcher [2, §1] showed that there exists a compact smooth manifold $\overline{M}^m$ satisfying

(1) $\mathrm{Int}(\overline{M}^m) = M^m$ and
(2) each component of $\partial \overline{M}^m$ is an almost flat Riemannian manifold.

Their construction is a consequence of concatenating results from [9], [3], and [11]. But Ruh [16] (extending results of Gromov [10]) showed that every almost flat manifold is an infranilmanifold.

Proof of Theorem 1. Let $\overline{M}^m$ be the compactification of M^m posited in Lemma 5. Note that $\partial \overline{M}^m$ is a π-manifold since M^m is parallelizable. Let N^{m-1} denote the boundary component of $\overline{M}^m$ corresponding to that cusp of M^m which contains the sets $f(t \times \mathbb{D}^{m-1})$ for all sufficiently large real numbers t. Hence, there clearly exists a compact smooth manifold W^m with the following properties:

(1) The interior of W^m is diffeomorphic to $M_{f,\phi}$.

(2) One of the components of ∂W^m is diffeomorphic to $N^{m-1} \# \Sigma^{m-1}$, where Σ^{m-1} is an exotic homotopy sphere representing the element of Θ_{m-1} determined by $\phi: S^{m-2} \to S^{m-2}$.

We now proceed via proof by contradiction; i.e., we assume that $M^m_{f,\phi}$ supports a complete pinched negatively curved Riemannian metric with finite volume. Consequently, Lemma 5 yields a second compactification of

$M_{f,\phi}$, i.e., a compact smooth manifold $\overline{M}^m_{f,\phi}$ satisfying properties (1) and (2) of the conclusion of Lemma 5 with M^m and $\overline{M}^m$ replaced respectively by $M^m_{f,\phi}$ and $\overline{M}^m_{f,\phi}$. But one easily sees that the boundaries of two smooth compactifications of the same manifold are smoothly h-cobordant. In particular, $N^{m-1}\#\Sigma^{m-1}$ is smoothly h-cobordant to an infranilmanifold. Hence $N^{m-1}\#\Sigma^{m-1}$ is diffeomorphic to an infranilmanifold because of [4], where it is shown that $\mathrm{Wh}\,\pi_1 N^{m-1} = 0$. But this contradicts Lemma 4 since N^{m-1} is a π-manifold. This completes the proof of Theorem 1.

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