ON 4-MANIFOLDS WITH FINITELY DOMINATED COVERING SPACES

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ABSTRACT. We show that if the universal covering space $\widetilde{M}$ of a closed 4-manifold M is finitely dominated then either M is aspherical, or $\widetilde{M}$ is homotopy equivalent to S^2 or S^3 , or $\pi_1(M)$ is finite. We also give a criterion for a closed 4-manifold to be homotopy equivalent to one which fibres over the circle.

INTRODUCTION

If M is the total space of a fibration with finitely dominated base B and fibre F then M is a Poincaré duality complex if and only if both B and F are (cf. [Go]). In particular, if $p : M_G \rightarrow M$ is a regular covering of a closed manifold M with covering group G then M_G is a Poincaré complex if it is finitely dominated and G is FP .

In this note we shall show that if M is a 4-manifold the latter conditions may be relaxed considerably. (The corresponding results are trivial for surfaces and are easy for 3-manifolds.) In Theorem 1 we show that M_G is a Poincaré complex if and only if it is finitely dominated and (if G has one end) its fundamental group is FP_3 . In Theorem 2 we give direct homological arguments to extend results of Hendriks and Laudenbach [HL] and Swarup [Sw] on the second homotopy groups of 3-manifolds to PD_3 -complexes with torsion free fundamental group. These results are used in Theorem 3, which considers infinite cyclic covers of 4-manifolds (i.e., $G \cong \mathbb{Z}$). Here we show that if $\pi_1(M_G)$ is virtually torsion free and is the fundamental group of a PD_3 -complex and $\text{Hom}_{\mathbb{Z}[\pi]}(\pi_2(M), \mathbb{Z}[\pi]) = 0$ (where $\pi = \pi_1(M)$) then M_G is a PD_3 -complex.

Our arguments rest on Poincaré duality and the Universal Coefficient spectral sequence (UCSS) for a regular covering $p : M_G \rightarrow M$ with covering group G . This spectral sequence has E_2^{pq} term $\text{Ext}_{\mathbb{Z}[G]}^q(H_p(M_G; \mathbb{Z}), \mathbb{Z}[G])$ and converges to $H^{p+q}(M; \mathbb{Z}[G])$. Similarly, we shall let LHSSS denote the Lyndon-Hochschild-Serre spectral sequence for an extension of groups. If G is a group and $w : G \rightarrow \{\pm 1\}$ is a homomorphism then $\bar{g} = w(g)g^{-1}$ defines an involution on $\mathbb{Z}[G]$. If L is a left $\mathbb{Z}[G]$ -module then $\bar{L}$ shall denote the conjugate

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right $Z[G]$ -module with the same underlying abelian group and $Z[G]$ -action given by $l.g = \bar{g}.l$, for all $l \in L$ and $g \in G$. (We shall also use the overbar to denote the conjugate of a right $Z[G]$ -module.) The conjugate of a free left (right) module is a free right (left) module of the same rank.

A group G is FP_n if the augmentation $Z[G]$ -module Z has a projective resolution which is finitely generated in degrees $\leq n$. It is FP if it has finite cohomological dimension and is FP_n for $n = \text{c.d. } G$. "Finitely generated" is equivalent to FP_1 , while "finitely presentable" implies FP_2 . Groups which are FP_2 are also said to be *almost finitely presentable*.

A PD_n -complex is a Poincaré duality complex of formal dimension n , in other words, a finitely dominated cell complex which satisfies Poincaré duality with arbitrary local coefficients. As such spaces are finitely dominated they have finitely presentable fundamental groups.

1. FINITELY DOMINATED COVERING SPACES

In this section we shall show that if a closed 4-manifold has an infinite regular covering space which is finitely dominated then the manifold is either aspherical or is close to being the total space of a fibre bundle.

Theorem 1. *Let M be a closed 4-manifold with fundamental group π , and let $\Pi = \pi_2(M)$. Suppose that $p : M_G \rightarrow M$ is a regular covering map, with covering group $G = \text{Aut}(p)$ and such that M_G is finitely dominated. Then*

- (i) G has finitely many ends;
- (ii) if M_G is acyclic then it is contractible and M is aspherical;
- (iii) if G has one end and $\pi_1(M_G)$ is infinite and FP_3 then M is aspherical and M_G is homotopy equivalent to S^1 or to an aspherical closed surface;
- (iv) if G has one end and $\pi_1(M_G)$ is finite but M_G is not acyclic then $M_G \simeq S^2$ or RP^2 ;
- (v) G has two ends if and only if M_G is a PD_3 -complex.

Proof. We may clearly assume that G is infinite and M is orientable. As $Z[G]$ has no nonzero (left) ideal (i.e., submodule) which is finitely generated as an abelian group $\text{Hom}_G(H_p(M_G; Z), Z[G]) = 0$ for all $p \geq 0$, the bottom row of the UCSS for the covering p is 0. From the UCSS and Poincaré duality we find that $H^1(G; Z[G]) \cong \bar{H}_3(M_G; Z)$. As this group is finitely generated and G is infinite, G has one or two ends.

Suppose that G has one end. Then $H_3(M_G; Z) = H_4(M_G; Z) = 0$. If M_G is acyclic then G is a PD_4 -group, so M_G is a PD_0 -complex and, hence, contractible, by [Go]. Hence M is aspherical.

If $\pi_1(M_G)$ is FP_3 then $\Pi = \pi_2(M) = \pi_2(M_G)$ is finitely generated as a module over $Z[\pi_1(M_G)]$, so $\text{Hom}_\pi(\Pi, Z[\pi]) = 0$. If, moreover, $\pi_1(M_G)$ is infinite then $H^s(\pi; Z[\pi]) = 0$ for $s \leq 2$. Hence $\Pi = 0$, by the UCSS and Poincaré duality, so M is aspherical. We may then apply an LHSSS corner argument to show that if $H^2(G; Z[G]) \neq 0$ then $\pi_1(M_G)$ has one end and $H^2(\pi_1(M_G); Z[\pi_1(M_G)]) \cong Z$. Since $\pi_1(M_G)$ is FP , it is a PD_2 -group by [Fa2, Theorem 3], so M_G is homotopy equivalent to an aspherical closed surface. If $H^2(G; Z[G]) = 0$ then $H^1(\pi_1(M_G); Z[\pi_1(M_G)]) \cong Z$ by another LHSSS corner argument. Since $\pi_1(M_G)$ is FP and hence torsionfree, it must be infinite cyclic. Therefore, $M_G \simeq S^1$.

If $\pi_1(M_G)$ is finite but M_G is not acyclic then the universal covering space $\tilde{M}$ is also finitely dominated but not contractible and $\Pi = H_2(\tilde{M}; Z)$ is a nontrivial finitely generated abelian group, while $H_3(\tilde{M}; Z) = H_4(\tilde{M}; Z) = 0$. Moreover, $H^2(\pi; Z[\pi]) \cong \Pi$ by the UCSS and Poincaré duality. Let C be a finite cyclic subgroup of π which acts trivially on Π . Then it follows easily from the Cartan-Leray spectral sequence for the projection of $\tilde{M}$ onto $\tilde{M}/C$ that there are isomorphisms $H_{n+3}(C; Z) \cong H_n(C; \Pi)$ for all $n > 2$. If n is odd then this isomorphism reduces to $0 = \Pi/|C|\Pi$. Since Π is finitely generated, this implies that multiplication by $|C|$ is an isomorphism. On the other hand, if n is even then we have $Z/|C|Z \cong \{a \in \Pi : |C|a = 0\}$. Hence we must have $C = 1$. Now since Π is finitely generated, any torsion subgroup of $\text{Aut}(\Pi)$ is finite. (Let T be the torsion subgroup of Π , and suppose that $\Pi/T \cong Z'$. Then the natural homomorphism from $\text{Aut}(\Pi)$ to $\text{Aut}(\Pi/T)$ has finite kernel, and its image is isomorphic to a subgroup of $\text{GL}(r, Z)$, which is virtually torsion free.) Hence as π is infinite, it must have elements of infinite order, so $H^2(\pi; Z[\pi])$ must be infinite cyclic by [Fa1, Corollary 5.2], as it is a nontrivial finitely generated group. Hence $\tilde{M} \simeq S^2$ and $\pi_1(M_G)$ has order at most 2, so $M_G \simeq S^2$ or RP^2 .

Suppose now that G has two ends. We may assume that $G \cong Z$, and it then follows easily from [Go] that M_G is a PD_3 -complex. Conversely, if M_G is a PD_3 -complex then G is infinite and $H^1(G; Z[G]) \cong H_3(M_G; Z)$, so G has two ends. $\square$

Is the hypothesis in (iii) that $\pi_1(M_G)$ be FP_3 redundant?

Corollary 1. *The covering space M_G is homotopy equivalent to a closed surface if and only if it is finitely dominated, G has one end, $H^2(G; Z[G]) \cong Z$, and $\pi_1(M_G)$ is FP_3 . The group G is then FP_∞ . If, moreover, G has a torsion free subgroup of finite index with infinite abelianization then M has a finite covering space which is simple homotopy equivalent to a closed 4-manifold which fibres over an aspherical closed surface.*

Proof. If M_G is homotopy equivalent to a closed surface then it is finitely dominated, $\pi_1(M_G)$ is FP_3 , and the conditions on G in the first assertion follow from the UCSS. The converse follows from parts (iii) and (iv) of the theorem.

If $\tilde{M} \simeq S^2$ and π is FP_n , we may splice together a cellular chain complex for $\tilde{M}$ and a finite partial resolution of the augmentation module Z to show that π is FP_{n+3} . (Since M is homotopy equivalent to a finite cell complex, the singular chain complex of $\tilde{M}$ is chain homotopy equivalent to a finite free $Z[\pi]$ -complex C_* . Let P_* be a projective resolution of the augmentation module Z in which $P_0 = Z[\pi]$ and P_i is finitely generated for all $i \leq n$. Let $Q_n = C_n$ if $n \leq 2$ and $Q_n = C_n \oplus P_{n-3}$ if $n \geq 3$, where the differential from Q_3 to Q_2 maps a generator of the summand P_0 onto a 2-cycle representing the generator of $H_2(\tilde{M}; Z)$.) An induction on n then shows that π is FP_∞ . If M is aspherical then π is a PD_4 -group and so is FP . Since finite groups are FP_∞ and surface groups are FP , it follows that π and $\pi_1(M_G)$ are both FP_∞ in all cases. Hence, so is $G \cong \pi/\pi_1(M_G)$ by [B, Proposition 2.7].

Suppose finally that H is a subgroup of finite index in G which is torsion free and has infinite abelianization. Then H is also finitely presentable, and

$H^2(H; Z[H]) \cong Z$. Therefore, H has finite cohomological dimension by [Fa1, Theorem 2.3] and so is a PD_2 -group by [Fa2]. The finite covering space $M_1 = M_G/H$ is simple homotopy equivalent to the total space of an M_G -bundle over the aspherical closed surface with fundamental group H by [Hi3, Theorems 1, 3, and 4]. $\square$

Does $H^2(G; Z[G]) \cong Z$ always imply that G is virtually a surface group? Is M in fact finitely covered by such a bundle space?

Corollary 2. *The covering space M_G is homotopy equivalent to S^1 if and only if it is finitely dominated, G has one end, $H^2(G; Z[G]) = 0$ and $\pi_1(M_G)$ is a nontrivial finitely generated free group. Moreover, M is homotopy equivalent to the total space of an S^1 -bundle over a PD_3 -complex if and only if G has finite cohomological dimension.*

Proof. If $M_G \simeq S^1$ then it is finitely dominated, M is aspherical, and the conditions on G in the first assertion follow from the LHSSS. The converse follows from part (iii) of the theorem, since a nontrivial finitely generated free group is infinite and FP . The second assertion follows from [Hi4]. $\square$

Corollary 3. *If the universal covering space $\tilde{M}$ is finitely dominated then either M is aspherical, or $\tilde{M}$ is homotopy equivalent to S^2 or S^3 , or $\pi_1(M)$ is finite.*

If $\tilde{M}$ is homotopy equivalent to S^2 and π has a subgroup of finite index with infinite abelianization then $\tilde{M}$ is homeomorphic to $S^2 \times R^2$ [Hi5].

We may ask how far the results of this section may be extended to arbitrary coverings. If $p: \hat{M} \rightarrow M$ is an infinite covering with $\hat{M}$ finitely dominated then must we have either M is aspherical, or $\tilde{M}$ is homotopy equivalent to S^2 , or $\hat{M}$ is a PD_3 -complex and $[\pi: N_\pi(\hat{\pi})]$ is finite?

2. THE ALGEBRAIC 2-TYPE OF A PD_3 -COMPLEX

In this section we shall use Schanuel's lemma to obtain some information on the algebraic 2-type of a PD_3 -complex with torsion free fundamental group, which we shall use in the next section. (We prove more than we need.)

Theorem 2. *Let N be a PD_3 -complex with torsionfree fundamental group ν . Then*

- (i) $\pi_2(N)$ is finitely generated as a $Z[\nu]$ -module;
- (ii) if ν is a nontrivial free group then $H^2(N; \pi_2(N)) \cong Z$;
- (iii) if ν is not free then $\pi_2(N)$ is a projective $Z[\nu]$ -module and $H^2(N; \pi_2(N)) = 0$;

(iv) if ν is not free any two of the conditions “ ν is FF ”, “ N is homotopy equivalent to a finite complex”, and “ $\pi_2(N)$ is stably free” imply the third.

Proof. We may clearly assume that $\nu \neq 1$. The PD_3 -complex N is homotopy equivalent to a connected sum of aspherical PD_3 -complexes and a 3-manifold with free fundamental group by [Tu, Theorem 2]. Therefore, ν is a corresponding free product, so it has cohomological dimension at most 3 and is FP .

Since N is finitely dominated, the equivariant chain complex of the universal covering space $\tilde{N}$ is chain homotopy equivalent to a complex $0 \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0$ of finitely generated projective left $Z[\nu]$ -modules. Then the sequences $0 \rightarrow Z_2 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow Z \rightarrow 0$ and $0 \rightarrow C_3 \rightarrow Z_2 \rightarrow \pi_2(N) \rightarrow 0$ are exact, where Z_2 is the module of 2-cycles in C_2 . Since ν

is FP and $\text{c.d. } \nu \leq 3$, Schanuel's lemma implies that Z_2 is projective and finitely generated. Hence $\pi_2(N)$ has projective dimension at most 1 and is finitely generated.

It follows easily from the UCSS and Poincaré duality that $\pi_2(N) \cong \bar{H}^1(\nu; Z[\nu])$ and that there is an exact sequence

$$H^3(\nu; Z[\nu]) \rightarrow H^3(N; Z[\nu]) \rightarrow \text{Ext}_\nu^1(\pi_2(N), Z[\nu]) \rightarrow 0.$$

The $w_1(N)$ -twisted augmentation homomorphism from $Z[\nu]$ to $\bar{Z}$ which sends $g \in \nu$ to $w_1(N)(g)$ induces an isomorphism from $H^3(N; Z[\nu])$ to $H^3(N; \bar{Z}) \cong Z$.

If ν is free then the first term in this sequence is 0, so $\text{Ext}_\nu^1(\pi_2(N), Z[\nu]) \cong Z$. (In particular, $\pi_2(N)$ has projective dimension 1.) There is also a short exact sequence of left modules $0 \rightarrow Z[\nu]^r \rightarrow Z[\nu] \rightarrow Z \rightarrow 0$, where r is the rank of ν . On dualizing we obtain the sequence of right modules $0 \rightarrow Z[\nu] \rightarrow Z[\nu]^r \rightarrow H^1(\nu; Z[\nu]) \rightarrow 0$. The long exact sequence of homology with these coefficients includes an exact sequence

$$\begin{aligned} 0 \rightarrow H_1(N; H^1(\nu; Z[\nu])) &\rightarrow H_0(N; Z[\nu]) \\ &\rightarrow H_0(N; Z[\nu]^r) \rightarrow H_0(N; H^1(\nu; Z[\nu])), \end{aligned}$$

in which the middle map is 0, so $H_1(N; H^1(\nu; Z[\nu])) \cong H_0(N; Z[\nu]) = Z$. Poincaré duality then gives

$$H^2(N; \pi_2(N)) \cong H_1(N; \pi_2(N)) = H_1(N; H^1(\nu; Z[\nu])) \cong Z.$$

If ν is *not* free then the map $H^3(\nu; Z[\nu]) \rightarrow H^3(N; Z[\nu])$ in the above sequence is onto, as can be seen by comparison with the corresponding sequence with coefficients $\bar{Z}$. Therefore, $\text{Ext}_\nu^1(\pi_2(N), Z[\nu]) = 0$. Since $\pi_2(N)$ has projective dimension at most one, it follows that $\pi_2(N)$ is in fact projective. As $H^2(N; Z[\nu]) = H_1(N; Z[\nu]) = 0$, it follows that $H^2(N; P) = 0$ for any projective $Z[\nu]$ -module P . Therefore, $H^2(N; \pi_2(N)) = 0$.

The final assertion follows easily from the fact that if $\pi_2(N)$ is projective then $Z_2 \cong \pi_2(N) \oplus C_3$. $\square$

If ν is not torsion free then the projective dimension of $\pi_2(N)$ is infinite. Swarup showed that if N is a 3-manifold which is the connected sum of a 3-manifold whose fundamental group is free of rank r with $s \geq 1$ aspherical 3-manifolds then $\pi_2(N)$ is a finitely generated free $Z[\nu]$ -module of rank $2r+s-1$ [Sw].

The cohomology group $H^2(N; \pi_2(N))$ arises in studying homotopy classes of self-homotopy equivalences of N . Hendriks and Laudenbach [HL] show that if N is a P^2 -irreducible 3-manifold and $\pi_1(N)$ is virtually free then $H^2(N; \pi_2(N)) \cong Z$; otherwise $H^2(N; \pi_2(N)) = 0$. Does this result extend to PD_3 -complexes with virtually torsion free fundamental group?

3. MAPPING TORI

An infinite cyclic covering space of a closed n -manifold M is homotopy equivalent to a PD_{n-1} -complex if and only if it is finitely dominated. The fundamental group ν of the covering space must then be finitely generated and $\chi(M) = 0$. It is conceivable that these conditions may suffice when $n = 4$; this is also so if either ν is free or ν does not contain the centre of $\pi_1(M)$ [Hi1]. Here we shall give another criterion.

Theorem 3. *Let M be a closed 4-manifold whose fundamental group π is an extension of Z by a virtually torsion free normal subgroup ν , and let $\Pi = \pi_2(M)$. Then the infinite cyclic covering space M' with fundamental group ν is homotopy equivalent to a PD_3 -complex if and only if ν is the fundamental group of a PD_3 -complex N , $\Pi^* = \text{Hom}_\pi(\Pi, Z[\pi])$ is 0 and the images of $k_1(M)$ and $k_1(N)$ in $H^3(\nu_0; \pi_2(M)) \cong H^3(\nu_0; \pi_2(N))$ generate the same subgroup under the action of $\text{Aut}_{Z[\nu_0]}(\pi_2(N))$, where ν_0 is a torsion free subgroup of finite index in ν . If, moreover, ν is torsion free and $w_1(M)|_\nu = w_1(N)$ then $M' \simeq N$, and if N is a 3-manifold whose irreducible factors are Haken, hyperbolic, or Seifert-fibred then M is homotopy equivalent to the total space of an N -bundle over S^1 .*

Proof. Suppose first that ν is torsion free and $w_1(M)|_\nu = w_1(N)$. If $\nu = 1$ then M' is simply connected and $\pi \cong Z$ has two ends. It follows immediately from the UCSS and Poincaré duality that $H_2(M'; Z) = \Pi \cong \bar{\Pi}^*$ and $H_3(M'; Z) \cong Z$. Therefore, M' is a PD_3 -complex if and only if $\Pi^* = 0$, and it is then homotopy equivalent to S^3 .

If ν is torsion free but nontrivial then π has one end and N is homotopy equivalent to a connected sum of aspherical PD_3 -complexes and a 3-manifold with free fundamental group [Tu]. Therefore, $\text{c.d. } \nu \leq 3$ and $\text{c.d. } \pi \leq 4$. It follows immediately from the UCSS that $e_\pi^s(\Pi) = 0$ if $s > 2$. Thus Π has projective dimension at most 2 as a $Z[\pi]$ -module and a fortiori as a $Z[\nu]$ -module. Hence $H^3(\nu; \Pi) = 0$. Moreover, the UCSS and Poincaré duality together with the hypothesis that $\Pi^* = 0$ imply that $\Pi \cong \bar{H}^2(\pi; Z[\pi])$. Hence $\Pi \cong \bar{H}^1(\nu; Z[\nu])$ by the LHSSS. (The overbar notation is unambiguous since $w_1(M)|_\nu = w_1(N)$.) But this is isomorphic to $\pi_2(N)$ by Poincaré duality for N . Since the first k -invariants of M' and N are equivalent, these spaces have the same algebraic 3-type. Since N is homotopy equivalent to a 3-dimensional complex, there is a map $f: N \rightarrow M'$ which induces isomorphisms on fundamental group and second homotopy group. Since the homology of the universal covering spaces of these spaces vanishes above degree 2, the map f is a homotopy equivalence.

Conversely, if M' is homotopy equivalent to N then $\Pi \cong \bar{H}^1(\nu; Z[\nu])$ is finitely generated as a $Z[\nu]$ -module. As 0 is the only $Z[\pi]$ -submodule of $Z[\pi]$ which is finitely generated as a $Z[\nu]$ -module, it follows that $\Pi^* = 0$.

In general, let $\nu_0 \subseteq \nu_+ \cap \pi_+$ be a torsion free subgroup of finite index in ν , where $\pi_+ = \ker w_1(M)$ and $\nu_+ = \ker w_1(N)$, and let $t \in \pi$ generate π modulo ν . Then each of the conjugates $t^k \nu_0 t^{-k}$ in π has the same index in ν . Since ν is finitely generated, the intersection $\mu = \bigcap t^k \nu_0 t^{-k}$ of all such conjugates has finite index in ν and is clearly torsion free and normal in the subgroup ρ generated by μ and t . It is easily seen that M' is a PD_3 -complex if and only if the covering space with fundamental group μ is. Moreover, $\text{Hom}_\rho(\Pi, Z[\rho]) = 0$ if and only if $\Pi^* = 0$. (For if $\{r_i\}$ is a transversal for ρ in π and $f: \Pi \rightarrow Z[\rho]$ is a nontrivial $Z[\rho]$ -linear homomorphism then $g(m) = \sum r_i f(r_i^{-1} m)$ defines a nontrivial element of Π^* . The converse is clear.)

Now suppose that there is such a homotopy equivalence $f: N \rightarrow M'$, where N is a 3-manifold whose irreducible factors are as above. Let $t: M' \rightarrow M'$ be the generator of the covering transformations. Then there is a self-homotopy

equivalence $u : N \rightarrow N$ such that $fu \sim tf$. As each irreducible factor of N has the property that self-homotopy equivalences are homotopic to homeomorphisms [Mo, Sc, Wd], u is homotopic to a homeomorphism [HL], so M is homotopy equivalent to the mapping torus of this homeomorphism. $\square$

All known PD_3 -complexes are homotopy equivalent to connected sums of Haken, hyperbolic or Seifert-fibred 3-manifolds, and PD_3 -complexes with finite fundamental group and, in particular, have virtually torsion free fundamental group.

The condition $\Pi^* = 0$ is often implied by the simpler condition $\chi(M) = 0$. For instance, this is so if ν is free (see [H, Chapter VII, Theorem 4]), so Theorem 3 gives an alternative proof of [Hi1, Theorem 1]. The condition $\chi(M) = 0$ also implies $\Pi^* = 0$ if π is amenable or $Z[\pi]$ embeds in a ring with the SIBN property which is flat as a $Z[\pi]$ -module and in which an element of the augmentation ideal becomes invertible, by [Ec] and [Hi2], respectively. (A ring R has the SIBN (stably invariant basis number) property if nontrivial stably free R -modules have well-defined, strictly positive rank—in other words, if whenever an R -module L satisfies $L \oplus R^a \cong R^b$ for some nonnegative integers a, b , then $b - a$ depends only on L and is 0 if and only if $L = 0$.)

It would also be of interest to replace the condition that ν be isomorphic to the group of a PD_3 -complex by more intrinsic, algebraic conditions. For instance, if $\chi(M) = 0$, π has an infinite cyclic normal subgroup, and ν is FP_2 then M' is a PD_3 -complex. (Note that here we do not assume a priori that ν is the fundamental group of a PD_3 -complex). This is [Hi1, Theorem 2] if the cyclic normal subgroup is not contained in ν and follows from the argument of [H, Chapter IV, Theorem 5] otherwise.

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