A CORRESPONDENCE THEOREM FOR MODULES OVER HOPF ALGEBRAS

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ABSTRACT. Let H be a finite-dimensional Hopf algebra. We prove that if M is a faithful H -module and if $H_1 \neq H_2$ are sub-Hopf algebras of H , then $M^{H_1} \neq M^{H_2}$, where M^{H_1} and M^{H_2} are the invariants in M under the respective actions of H_1 and H_2 . We also show that if $H_1 \neq H_2$, then H_1 and H_2 have different left integrals. Both of these results rely heavily on the freeness theorem of Nichols-Zoeller.

In studying the action of a Hopf algebra H on an algebra A , one is interested in determining when there exists a Galois correspondence between the sub-Hopf algebras of H and certain subalgebras of A . In particular, it is of interest to know when the sub-Hopf algebras of H all have different subalgebras of invariants. Along these lines, the main result of this paper is

Theorem. *Let H be a finite-dimensional Hopf algebra with a faithful module M . If $H_1 \neq H_2$ are sub-Hopf algebras of H , then $M^{H_1} \neq M^{H_2}$.*

Notice that the theorem does not require that M be an algebra or an H -module algebra, merely that M be an H -module. Some examples of faithful actions of Hopf algebras on algebras are when A is an H -module algebra such that the smash product $A \# H$ is simple or when A is a prime H -module algebra and H is either the group algebra of a group of "outer" automorphisms or the restricted enveloping algebra of a Lie algebra of "outer" derivations. For further details on smash products, outer automorphisms, and outer derivations, see [BM].

We now fix the notation we will use throughout this paper. H will denote a finite-dimensional Hopf algebra over a field k with comultiplication Δ , counit ϵ , and antipode S . If K is a sub-Hopf algebra of H then the left integral of K will be denoted as $\int_K$ and we let $\omega_K = K \cap \ker \epsilon$. By a result of Larson-Sweedler [LS], $\int_K$ is a one-dimensional ideal of K and is the right annihilator in K of ω_K . If M is an H -module, we let $M^K = \{m \in M \mid h \cdot m = \epsilon(h)m, \text{ for all } h \in K\} = \{m \in M \mid h \cdot m = 0, \text{ for all } h \in \omega_K\}$ denote the invariants in M under the action of K .

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If H_1 and H_2 are sub-Hopf algebras of H , let $K = \langle H_1, H_2 \rangle$ be the subalgebra of H generated by H_1 and H_2 . We recall that Δ and S are, respectively, an algebra homomorphism and an algebra anti-homomorphism. Since K consists solely of sums of products of elements of H_1 and H_2 , it follows that $\Delta(K) \subseteq K \otimes K$ and $S(K) \subseteq K$. Thus K is also a sub-Hopf algebra of H .

Lemma 1. *If M is an H -module and H_1 and H_2 are sub-Hopf algebras of H , then $M^K = M^{H_1} \cap M^{H_2}$, where $K = \langle H_1, H_2 \rangle$.*

Proof. Since K contains both H_1 and H_2 , it is clear that M^K is contained in both M^{H_1} and M^{H_2} . Thus $M^K \subseteq M^{H_1} \cap M^{H_2}$. Now suppose $m \in M^{H_1} \cap M^{H_2}$ and $h = h_1 h_2 \cdots h_n$, where each $h_i \in H_1 \cup H_2$. Since ϵ is an algebra homomorphism, we have

$$\begin{aligned} h \cdot m &= (h_1 h_2 \cdots h_n) \cdot m = (h_1 h_2 \cdots h_{n-1}) \cdot (h_n \cdot m) \\ &= (h_1 h_2 \cdots h_{n-1}) \cdot \epsilon(h_n) m = \cdots = \epsilon(h_1) \epsilon(h_2) \cdots \epsilon(h_n) m \\ &= \epsilon(h_1 h_2 \cdots h_n) m = \epsilon(h) m. \end{aligned}$$

If $h \in K$ then we can write $h = a_1 + a_2 + \cdots + a_s$, where each a_i is a product of elements in $H_1 \cup H_2$. Therefore, our previous argument shows that

$$\begin{aligned} h \cdot m &= (a_1 + a_2 + \cdots + a_s) \cdot m = a_1 \cdot m + a_2 \cdot m + \cdots + a_s \cdot m \\ &= \epsilon(a_1) m + \epsilon(a_2) m + \cdots + \epsilon(a_s) m = \epsilon(a_1 + a_2 + \cdots + a_s) m = \epsilon(h) m. \end{aligned}$$

Thus $m \in M^K$ and we have shown the reverse inclusion. $\square$

We next use the Nichols-Zoeller [NZ] freeness result to prove a lemma on left integrals.

Lemma 2. *If H is a finite-dimensional Hopf algebra and if $H_1 \neq H$ is a sub-Hopf algebra, then $\int_{H_1} \neq \int_H$.*

Proof. By way of contradiction, suppose there exists some $0 \neq t \in \int_{H_1}$ such that $t \in \int_H$. The freeness result of Nichols-Zoeller states that H is free both as a left and right H_1 -module. Thus there exists $e_1, e_2, \dots, e_n$ such that $H = H_1 e_1 \oplus H_1 e_2 \oplus \cdots \oplus H_1 e_n$. Since the left integral of H is a one-dimensional ideal of H , we have $te_1 = \lambda_1 t$ and $te_2 = \lambda_2 t$ for some $\lambda_1, \lambda_2 \in k$. Since $t \in H_1$, it follows that both λ_1 and λ_2 are nonzero. Thus $(\lambda_2 t) e_1 - (\lambda_1 t) e_2 = (\lambda_1 \lambda_2 - \lambda_1 \lambda_2) t = 0$, contradicting that e_1 and e_2 are part of a free basis for H over H_1 . As a result, $\int_{H_1} \neq \int_H$. $\square$

We now combine the previous two lemmas to prove the theorem we mentioned at the outset of this paper.

Theorem 3. *Let H be a finite-dimensional Hopf algebra with a faithful module M . If $H_1 \neq H_2$ are sub-Hopf algebras of H , then $M^{H_1} \neq M^{H_2}$.*

Proof. Suppose not; therefore we will assume that $M^{H_1} = M^{H_2}$. By Lemma 1, if $K = \langle H_1, H_2 \rangle$, then $M^K = M^{H_1} = M^{H_2}$. However, if $H_1 \neq H_2$, then K must properly contain either H_1 or H_2 . Without loss of generality, we may assume that $H_1 \neq K$, thus we have reduced to the case where M is a faithful K -module and H_1 is a proper sub-Hopf algebra of K such that K and H_1 have the same set of invariants when acting on M .

Let $0 \neq t \in \int_{H_1}$; since $\omega_{H_1} t = 0$ it follows that $\omega_{H_1} \cdot (t \cdot M) = 0$. Thus $t \cdot M \subseteq M^{H_1} = M^K$. As a result $\omega_K \cdot (t \cdot M) \subseteq \omega_K \cdot M^K = 0$. Since K acts

faithfully on M , we have $\omega_K t = 0$, and so t belongs to the right annihilator in K of ω_K . Hence $t \in \int_K$ which contradicts Lemma 2, thereby proving the result. $\square$

We can also use Lemmas 1 and 2 to prove

Theorem 4. *Let H be a finite-dimensional Hopf algebra. If $H_1 \neq H_2$ are sub-Hopf algebras of H then $\int_{H_1} \neq \int_{H_2}$.*

Proof. We give H the structure of a left H -module M by letting H act on M via left multiplication. Note that for any sub-Hopf algebra J we have $M^J = \{m \in M \mid \omega_J \cdot m = 0\}$, therefore, $M^J \cap J = \int_J$. By way of contradiction, let $0 \neq t \in \int_{H_1} = \int_{H_2}$ and let $K = \langle H_1, H_2 \rangle$. Now, by Lemma 1, $t \in (M^{H_1} \cap H_1) \cap (M^{H_2} \cap H_2) \subseteq (M^{H_1} \cap M^{H_2}) \cap K = M^K \cap K = \int_K$. Thus $\int_K = \int_{H_1} = \int_{H_2}$. However, since K must properly contain either H_1 or H_2 , this contradicts Lemma 2, thereby proving the result. $\square$

In light of the above results, one might suspect that if $K = \langle H_1, H_2 \rangle$ then $\int_K$ is somehow built up from $\int_{H_1}$ and $\int_{H_2}$. However, this appears not to be the case. For example, suppose $G = S_n$, the symmetric group on n letters, and let g be an element of order 2 and h an element of order n such that g and h generate G . Let G_1 and G_2 be, respectively, the cyclic subgroups of G generated by g and h . Given a field k , let H be the group algebra kG , and let H_1 and H_2 be the sub-Hopf algebras kG_1 and kG_2 . In this case $\int_{H_1}$ is the subspace spanned by $1 + g$ and $\int_{H_2}$ is the subspace spanned by $1 + h + \cdots + h^{n-1}$. However, $H = \langle H_1, H_2 \rangle$ hence $\int_K$ is spanned by the sum of all $n!$ elements of G .

REFERENCES

- [BM] J. Bergen and S. Montgomery, *Smash products and outer derivations*, Israel J. Math. **53** (1986), 321–345.
- [LS] R. G. Larson and M. E. Sweedler, *An associative bilinear form for Hopf algebras*, Amer. J. Math. **91** (1969), 75–94.
- [NZ] W. D. Nichols and M. Bettina Zoeller, *A Hopf algebra freeness theorem*, Amer. J. Math. **111** (1989), 381–385.

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