AN EXAMPLE CONCERNING THE YOSIDA-HEWITT DECOMPOSITION OF FINITELY ADDITIVE MEASURES

WOLFGANG HENSGEN

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ABSTRACT. Let λ be Lebesgue measure on the Lebesgue σ -algebra $\mathcal{L}$ of $I :=]0, 1[$. The author gives an example of a purely finitely additive measure $\varphi: \mathcal{L} \rightarrow [0, 1]$ vanishing on λ -null sets such that $\int f d\varphi = \int f d\lambda$ for every bounded continuous function f on I ($f \in C_b(I)$). Consequently, $\lambda - \varphi \in L^\infty(\lambda)'$ annihilates $C_b(I)$ and is *not* purely finitely additive, contrary to an assertion of Yosida and Hewitt.

To be in accordance with [HY], the scalar field is $\mathbb{R}$, although $\mathbb{C}$ could be used with insignificant changes in what follows. Also, only *bounded* real-valued set functions are considered. A finitely additive measure $\varphi \geq 0$ on an algebra of sets is called purely finitely additive (p.f.a.) if every σ -additive measure μ , $0 \leq \mu \leq \varphi$, is zero. If the requirement $\varphi \geq 0$ is dropped, φ is called p.f.a. if the positive and negative variations φ_+ and φ_- are both p.f.a. Every finitely additive measure can be uniquely written as $\mu + \varphi$, where μ is σ -additive and φ is p.f.a. This is the content of the Yosida-Hewitt decomposition theorem [HY, 1.24]. In later sections of this fundamental work on finitely additive measures, the authors study the bounded linear functionals on $L^\infty(\lambda)$. Those are easily identified as the finitely additive measures on $\mathcal{L}$ *vanishing on λ -null sets* [HY, 2.3]. Such a measure φ is p.f.a. if and only if it is concentrated on sets of arbitrarily small λ -measure, where φ is called concentrated on $E \in \mathcal{L}$ if $|\varphi|(I \setminus E) = 0$ [HY, 3.1].

Let us turn to the construction of the example announced in the abstract. For any $t_0 \in I$ there exists a positive (hence bounded) linear functional φ_0 on $L^\infty(\lambda)$ with $\varphi_0|_{C_b(I)} = \delta_{t_0} :=$ point evaluation at t_0 . Just apply the Hahn-Banach theorem to find a linear functional below the sublinear functional $f \mapsto \text{ess lim sup}_{t \rightarrow t_0} f(t) := \lim_{\varepsilon \rightarrow 0} \text{ess sup}_{t_0 - \varepsilon \leq t \leq t_0 + \varepsilon} f(t)$ on $L^\infty(\lambda)$. [One could even find a character of $L^\infty(\lambda)$ extending δ_{t_0} (see, e.g., [IT, p. 107], but I will not need this.) Any such φ_0 is concentrated on every neighbourhood U of t_0 (in particular p.f.a.). To see this, take any $f \in C_b(I)$, $f \leq 1_U$, $f(t_0) = 1$. Then $1 = \delta_{t_0}(f) = \varphi_0(f) \leq \varphi_0(1_U) \leq \varphi_0(1) = 1$, so $\varphi_0(1_U) = 1$.

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This example is contained in the author's habilitation thesis [H].

Now let $t_{nj} := (j + \frac{1}{2})2^{-n}$ for $n \in \mathbb{N}$, $0 \leq j \leq 2^n - 1$, and φ_{nj} be a positive functional on $L^\infty(\lambda)$ with $\varphi_{nj}|_{C_b(I)} = \delta_{t_{nj}}$. Next, choose and fix a Banach limit Lim , that is a positive (hence bounded) linear extension of the “lim” functional over ℓ^∞ . (Take a linear functional below $\limsup$.) For $x = (x_n)_{n \in \mathbb{N}} \in \ell^\infty$ write $\text{Lim}_{n \rightarrow \infty} x_n$ instead of $\text{Lim}(x)$.

Define $\varphi(f) := \text{Lim}_{n \rightarrow \infty} 2^{-n} \sum_{j=0}^{2^n-1} \varphi_{nj}(f)$ for $f \in L^\infty(\lambda)$. Obviously, $\varphi \in L^\infty(\lambda)'$ is positive and concentrated on every neighbourhood of the set of points t_{nj} ; hence, φ is p.f.a. Moreover, any $f \in C_b(I)$ arbitrarily extended over $[0, 1]$ is Riemann integrable over $[0, 1]$, so $\varphi(f) = \int_0^1 f(t) dt = \int f d\lambda$. $\varphi(1)$ being 1, the set function $\varphi(A) = \varphi(1_A)$, $A \in \mathcal{L}$, has all the announced properties.

Finally, the difference $\lambda - \varphi \in L^\infty(\lambda)'$ annihilates $C_b(I)$ but is *not* p.f.a. (uniqueness of the decomposition!), contrary to the last assertion of [HY, Theorem 3.4].

The careful reader might argue that Yosida and Hewitt consider $\mathbb{R}$ instead of I . However, note that *their* measure λ is an (arbitrary) *probability* measure equivalent to Lebesgue measure on $\mathbb{R}$. Therefore, any both-way null set-preserving homeomorphism $I \rightarrow \mathbb{R}$ can be used for translation.

NOTED ADDED IN PROOF

After this note had been accepted for publication, I found that M. Valadier (*Une singulière forme linéaire singulière sur L^∞* , Sémin. d'Analyse Convexe Montpellier 1987, Exposé no. 4) has (also) constructed a p.f.a. functional on $L^\infty(\lambda)$ which extends the functional λ on $C[0, 1]$ (!). In this connection, see also the article by Y. A. Abramovich and A. W. Wickstead, *Singular extensions and restrictions of order continuous functionals*, Hokkaido Math. J. **21** (1992), 475–482.

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UNIVERSITÄT REGENSBURG, NATURWISSENSCHAFTLICHE FAKULTÄT I-MATHEMATIK, UNIVERSITÄTSSTRASSE 31, 93040 REGENSBURG, GERMANY

E-mail address: hensgen@alf2.ngate.uni-regensburg.dbp.de