ON UPPER SEMICONTINUITY OF DUALITY MAPPINGS

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ABSTRACT. We give new sufficient conditions for a Banach space to be an Asplund (or reflexive) space in terms of certain upper semicontinuity of the duality mapping.

1. INTRODUCTION, NOTATION, AND STATEMENT OF RESULTS

Recall that a Banach space X is said to be an Asplund space if every continuous, real-valued, convex function defined on an open subset of X is Fréchet-differentiable on a dense G_δ subset of its domain—a notion introduced by Asplund [2] under the name “Strong Differentiability Space”. Intensive research on Asplund spaces culminated in Stegall’s Theorems [21], showing that X is an Asplund space if and only if X^* has the Radon-Nikodym property, if and only if every separable subspace of X has a separable dual. Simple geometrical conditions already implying that a space is Asplund also have a long tradition. Asplund already showed that X is an Asplund space if X^* is locally uniformly rotund—a result which was improved by Ekeland and Lebourg [7] by showing that a Banach space on which there exists a Fréchet-differentiable real function with bounded nonempty support is an Asplund space; in particular, a Banach space with *Fréchet-differentiable* norm is Asplund. In view of Stegall’s Theorem, the last assertion also follows from the result by Diestel and Faires [6] that X^* has the Radon-Nikodym property whenever X has a *very smooth* norm. Before recalling this notion, let us consider the *duality mapping* of a Banach space X , namely, the set-valued mapping D from the unit sphere S_X of X into the subsets of S_{X^*} given by

$$D(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}.$$

The norm of X (or X itself) is said to be *smooth* when D is single valued and *very smooth* if, in addition, D is continuous for the norm topology of X and the weak topology of X^* . Note that the norm of X is Fréchet-differentiable on S_X if and only if D is single valued and continuous for the norm topologies of X and X^* ; hence every Fréchet-differentiable norm is very smooth.

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In the converse direction, the question if every Asplund space has an equivalent Fréchet-differentiable norm remained open for a long time and was recently answered in the negative by Haydon [17], who found an Asplund space with no equivalent smooth norm! In view of Haydon's counterexample, it now seems more interesting to look for weaker sufficient conditions for a space to be Asplund, specially those not implying smoothness.

In the absence of smoothness it is natural to deal with "continuity" properties of the set-valued mapping D . Let us first recall the notions of upper and lower semicontinuity for set-valued mappings which are usually handled in topology. A set-valued mapping F between topological spaces S and T is said to be upper (resp. lower) semicontinuous at a point $x \in S$ if for every open subset Ω of T such that $F(x) \subseteq \Omega$ (resp. $F(x) \cap \Omega \neq \emptyset$) there is a neighborhood U of x such that $F(y) \subseteq \Omega$ (resp. $F(y) \cap \Omega \neq \emptyset$) for $y \in U$. We take for S the unit sphere of our Banach space which will always be provided with the norm topology, and on $T = X^*$ we may consider either the weak-*, the weak, or the norm topology. Lower semicontinuity is useless for our purposes, for if the duality mapping D is lower semicontinuous for the weak-* topology of X^* at a point $x \in S_X$, then $D(x)$ is a singleton (an observation due to Cudia [3]) and we are back in the smooth case. On the other hand, Cudia also showed that D is always upper semicontinuous on S_X if we consider the weak-* topology on X^* and that by taking either the weak or the norm topology we get natural generalizations of very smooth and Fréchet-differentiable norms respectively. Hu and Lin [18] have shown that X is an Asplund space whenever D is upper semicontinuous on S_X for the weak topology of X^* , extending the above-mentioned result by Diestel and Faires.

Giles, Gregory, and Sims introduced in [11] a weaker form of upper semicontinuity which is specially suitable for dealing with the duality mapping. Given a locally convex topology τ on X^* they consider the following kind of continuity of the duality mapping D at a point $x \in S_X$:

- (*) For every τ -neighborhood of zero V on X^* there is a $\delta > 0$ such that $D(y) \subseteq D(x) + V$ whenever $y \in S_X$ satisfies $\|y - x\| < \delta$.

Thus, instead of an arbitrary τ -open set in X^* containing $D(x)$, they restrict attention to sets of the form $D(x) + V$, where V is a τ -neighborhood of zero. It is clear that (*) holds whenever D is upper semicontinuous at x for the topology τ and both properties are equivalent in case $D(x)$ is τ -compact. It was shown in [11] that if (*) holds for all $x \in S_X$, τ being the weak topology, and either $D(x)$ is weakly compact for all x or there is a constant $k < 1$ such that the diameter of $D(x)$ is less than k for all x in S_X , then X is an Asplund space. The first of these results is a particular case of the above-mentioned (later) result by Hu and Lin.

The authors of [11] refer to condition (*) as $(n - \tau)$ -upper semicontinuity of D at x , a terminology which might be confusing, so we shall try to avoid it. Gregory [16] showed that, when τ is the norm topology, condition (*) holds if and only if the norm is *strongly subdifferentiable* at x ; that is, the radial limit

$$\lim_{t \rightarrow 0+} \frac{\|x + ty\| - 1}{t}$$

exists uniformly for $y \in S_X$. This terminology was introduced in [10], where an account of strong subdifferentiability is presented. The paper [10] also contains the important result by Godefroy (see also [13]) that a Banach space whose norm is strongly subdifferentiable at all points of its unit sphere is an Asplund space. Thus there are three nonsmooth geometrical conditions which are sufficient for a Banach space to be Asplund. All of them clearly imply that $(*)$ holds for all $x \in S_X$ when we take for τ the weak topology of X^* , which is the property we deal with in this paper. We introduce the following terminology.

1.1. Definition. We say that the norm of a Banach space X is *quite smooth* at a point $x \in S_X$ if for every neighborhood of zero V in the weak topology of X^* there is a $\delta > 0$ such that

$$D(y) \subseteq D(x) + V$$

whenever $y \in S_X$ satisfies $\|y - x\| < \delta$. We may also say that X is *quite smooth* if its norm is quite smooth at all points of the unit sphere.

The first main result in this paper is the affirmative answer to the question posed in [11, Problem 2] and reads as follows.

1.2. Theorem. *Every quite smooth Banach space is an Asplund space.*

This theorem clearly implies the above-mentioned results by Hu and Lin, Giles, Gregory, and Sims, and Godefroy. In particular, it shows that the equivalent conditions appearing in [18, Theorem 9] are not only equivalent but true! Our proof consists of a slight refinement of the techniques developed by Godefroy for the proof of [10, Theorem 5.1]. Actually we only add a simple observation to Godefroy's ideas.

The same arguments also allow us to extend several known results on sufficient conditions for reflexivity of a dual Banach space. The fact that a dual Banach space with Fréchet-differentiable norm is reflexive is a well-known result, sometimes attributed to Smulian (see [4, 5]) but probably due to Fan and Glicksberg [8, Theorem 3]. Two generalizations of this result are also known. On the one hand, Diestel [5, Theorem 1, p. 33] shows that a very smooth dual Banach space is reflexive, and the proof can easily be modified to show that if the duality mapping of a dual Banach space X is upper semicontinuous for the weak topology of X^* , then X is reflexive. In particular, if X is a quite smooth dual Banach space whose duality mapping has weakly compact values, then X is reflexive—a result proved by Zhang [22, Theorem 2]. On the other hand, it was shown in [1] (see also [10, Theorem 3.3]) that a dual Banach space whose norm is strongly subdifferentiable on the unit sphere must be reflexive. This result has been improved by Godefroy [13], who shows that the assumption of being a dual space can be weakened by only assuming that it satisfies the so-called *finite-infinite intersection property* ($IP_{f,\infty}$ for short). A Banach space X is said to satisfy the $IP_{f,\infty}$ if every family of closed balls in X with empty intersection contains a finite subfamily with empty intersection. It is easy to show that if there is a norm-one projection from X^{**} onto X (in particular, if X is a dual space), then X satisfies the $IP_{f,\infty}$. Herein we prove

1.3. Theorem. *A Banach space X is reflexive if (and only if) it is quite smooth and satisfies the $IP_{f,\infty}$.*

This paper also contains some examples which clarify the relation between all the above-mentioned properties of the duality mapping. It is worth pointing out that we give an example of a quite smooth Banach space (even more, a space whose norm is strongly subdifferentiable) whose duality mapping is not upper semicontinuous for the weak topology of X^* . In particular, upper semicontinuity in the sense of Giles, Gregory, and Sims is strictly weaker than the usual notion—a fact which seems to be new. Our examples also show that the assumptions in Theorem 1.2 and 1.3 are strictly more general than those in all previous results which we have used for motivation.

2. PROOF OF THE MAIN RESULTS

The main tool in our proofs will be the following lemma, the so-called “Simons’s inequality”. We use a slightly modified form which is valid in the complex as well as in the real case. Its proof is the same as the original version.

2.1. Lemma (Simons [20, Lemma 2]). *Let E be an arbitrary nonempty set, and consider the Banach space l_∞^E of all bounded scalar-valued functions on E with its usual sup-norm. Let $\{x_n\}$ be a bounded sequence in l_∞^E , and suppose that a subset B of E satisfies the following condition: for any sequence $\{\alpha_n\}$ of positive numbers such that $\sum_{n=1}^\infty \alpha_n = 1$ there exists some $b \in B$ satisfying*

$$\left| \sum_{n=1}^\infty \alpha_n x_n(b) \right| = \left\| \sum_{n=1}^\infty \alpha_n x_n \right\|.$$

Then we have the inequality

$$\sup_{b \in B} \limsup_{n \rightarrow \infty} |x_n(b)| \geq \inf\{\|y\|: y \in \text{co}\{x_n: n \in \mathbb{N}\}\},$$

where “co” denotes convex hull.

The next lemma is the crucial step in the proof of Theorems 1.2 and 1.3. It is a slight improvement of a result by Godefroy [13, Lemma 4]. The unit ball of a Banach space X will be denoted by B_X .

2.2. Lemma. *Let B be a “boundary” for a Banach space X ; that is, $B \subseteq S_{X^*}$ and $B \cap D(x) \neq \emptyset$ for all $x \in S_X$. Assume that*

$$B \subseteq \overline{\text{co}}(F + \alpha B_{X^*})$$

for some countable set $F \subseteq X^$ and some $\alpha < 1$, where $\overline{\text{co}}$ stands for norm-closed convex hull. Then X^* is the closed linear span of F ; hence, X^* is separable.*

Proof. Assume on the contrary that F is contained in a proper norm-closed subspace of X^* , so the Hahn-Banach Theorem provides us with $x^{**} \in S_{X^{**}}$ such that $x^{**}(x^*) = 0$ for all x^* in F . We choose $\alpha < \beta < 1$ and $z^* \in S_{X^*}$ such that $\text{Re } x^{**}(z^*) > \beta$. Now we use the fact that B_X is dense in $B_{X^{**}}$ for the 1-countable topology of pointwise convergence on the set $F \cup \{z^*\}$ to find a sequence $\{x_n\}$ in B_X such that

$$(a) \quad \{x^*(x_n)\} \xrightarrow{n \rightarrow \infty} x^{**}(x^*) = 0 \quad \text{for all } x^* \in F$$

and $\{z^*(x_n)\} \rightarrow x^{**}(z^*)$. Without loss of generality we may assume that $\{x_n\}$ satisfies

$$(b) \quad \text{Re } z^*(x_n) \geq \beta \quad \text{for all } n.$$

Consider the real-valued function Φ defined on X^* by

$$\Phi(y^*) = \limsup_{n \rightarrow \infty} |y^*(x_n)| \quad (y^* \in X^*).$$

One can easily check that Φ is convex and lower semicontinuous for the norm topology of X^* .

In view of (a) we clearly have $\Phi(y^*) \leq \alpha$ for $y^* \in F + \alpha\beta_{X^*}$; hence our assumption on B gives

$$\sup\{\Phi(y^*): y^* \in B\} \leq \alpha.$$

Since B is a boundary for X we can apply Lemma 2.1 with $E = B_{X^*}$, and we get

$$\inf\{\|y\|: y \in \text{co}\{x_n: n \in \mathbb{N}\}\} \leq \alpha.$$

This is a contradiction, for (b) implies that

$$\|y\| \geq \text{Re } z^*(y) \geq \beta > \alpha$$

for $y \in \text{co}\{x_n: n \in \mathbb{N}\}$. $\square$

Recall that a norm-closed subspace N of the dual X^* of a Banach space X is said to be a *norming subspace* if it satisfies

$$\|x\| = \sup\{|x^*(x)|: x^* \in B_N\}$$

for all x in X . The proof of Theorem 1.2 will easily follow from

2.3. Lemma. *Let X be a quite smooth separable Banach space. Then X^* contains no proper norming subspace; hence, X^* is separable.*

Proof. Let N be a norming subspace of X^* , and let $x \in S_X$ be such that $D(x)$ is a singleton, say x^* . We choose a sequence $\{y_n^*\}$ in B_N such that $\{y_n^*(x)\} \rightarrow 1$ and apply [11, Theorem 2.1] to conclude that $\{y_n^*\}$ converges to x^* in the weak topology of X^* , so $x^* \in N$.

Now we apply Mazur's Theorem (see, for example, [19, Theorem 1.20]) to get a dense sequence $\{x_k\}$ in S_X such that $D(x_k)$ is a singleton, say x_k^* , for all k . Since $x_k^* \in N$ for all k , in view of Lemma 2.2 we are left with showing that B is a boundary for X , where

$$B = A \cap S_{X^*} \quad \text{with } A = \text{co}(F + \tfrac{1}{2}B_{X^*}) \quad \text{and } F = \{x_k^*: k \in \mathbb{N}\}.$$

Assume on the contrary that there is some $x \in S_X$ such that $B \cap D(x) = \emptyset$; that is, $A \cap D(x) = \emptyset$. Since A is convex with nonempty norm-interior and $D(x)$ is also convex, we can apply the Hahn-Banach Separation Theorem to find some $x^{**} \in S_{X^*}$ such that

$$\sup\{\text{Re } x^{**}(a): a \in A\} \leq \inf\{\text{Re } x^{**}(x^*): x^* \in D(x)\}$$

and the definition of A gives

$$\text{Re } x^{**}(x_k^*) + \tfrac{1}{2} \leq \inf\{\text{Re } x^{**}(x^*): x^* \in D(x)\};$$

hence,

$$|x^{**}(x_k^*) - x^{**}(x^*)| \geq \tfrac{1}{2}$$

for all $x^* \in D(x)$ and $k \in \mathbb{N}$. Now we apply that the norm of X is quite smooth at x by considering the weak neighborhood of zero in X^* given by

$$W = \{x^* \in X^*: |x^{**}(x^*)| < \frac{1}{2}\},$$

and we find a $\delta > 0$ such that $D(y) \subseteq D(x) + W$ whenever $y \in S_X$ satisfies $\|y - x\| < \delta$. It follows that $\|x_k - x\| \geq \delta$ for all k , a contradiction. Thus B is a boundary for X as required.

The fact that X^* is separable has already been shown, but note that if X is any separable Banach space, X^* always contains a separable norming subspace. $\square$

Proof of Theorem 1.2. It suffices to apply the above lemma and to take into account the elementary fact that a closed subspace of a quite smooth Banach space is also quite smooth (see [11, Lemma 1.1]). $\square$

As mentioned in the introduction, the results in [11, Corollary 3.3; 18, Theorem 10; 13, Proposition 8] are immediate consequences of Theorem 1.2. We will see later that these three results are independent.

Proof of Theorem 1.3. Let X be a quite smooth Banach space satisfying the $IP_{f,\infty}$. By Lemma 2.3 and the fact that being quite smooth is a hereditary property, we have that Y^* contains no proper norming subspace for any separable closed subspace Y of X . It follows from the results by Godefroy and Kalton on the so-called ball topology [14, Theorem II.4, Proposition II.5] that X^* itself contains no proper norming subspace. Together with the $IP_{f,\infty}$, this implies that X is reflexive by a result due to Godefroy (see [12, Theorem 3]). $\square$

Theorem 1.3 in turn improves the results in [5, Theorem 1, p. 33; 1, Corollary 3.5; 22, Theorem 2].

3. EXAMPLES

Let us compare the various properties of a Banach space X with the duality mapping D which have been considered in this paper. They are the following:

- (1) The norm of X is Fréchet differentiable on S_X .
- (2) X is very smooth.
- (3) D is upper semicontinuous for the weak topology of X^* .
- (4) D is upper semicontinuous for the norm topology of X^* .
- (5) X is quite smooth and $\text{diam}(D(x)) \leq k < 1$ for all x in S_X .
- (6) X is quite smooth.
- (7) The norm of X is strongly subdifferentiable on S_X .

The obvious relations between the above assertions are:

$$\begin{array}{ccccc} (1) & & & & (1) \\ \Downarrow & & & & \Downarrow \\ (2) & \Rightarrow & (3) & \Leftarrow & (4) \\ \Downarrow & & \Downarrow & & \Downarrow \\ (5) & \Rightarrow & (6) & \Leftarrow & (7) \end{array}$$

We intend to give examples showing that no other implication in the diagram is true. It is clearly enough to show that $(2) \not\Rightarrow (7)$, $(4) \not\Rightarrow (5)$, $(5) \not\Rightarrow (3)$, and $(7) \not\Rightarrow (3)$.

3.1. Examples. (a) Consider a reflexive Banach space which is smooth (hence very smooth) but whose norm is not Fréchet differentiable. Actually, it follows from a result by Franchetti [9, Proposition 1] that any reflexive Banach space can be renormed to satisfy these conditions. This shows that $(2) \not\Rightarrow (7)$.

(b) The space c_0 satisfies (4) but fails (5).

(c) We finally give an example of a Banach space satisfying (5) and (7) but failing (3). Let Y be a nonreflexive Banach space whose norm is Fréchet differentiable on the unit sphere (take, for example, c_0 with an equivalent Fréchet differentiable norm). Now we take $X = \mathbb{K} \times Y$ with the norm given by

$$\|(\lambda, y)\| = F(|\lambda|, \|y\|) \quad (\lambda \in \mathbb{K}, y \in Y),$$

where F is the norm on $\mathbb{R}^2$ defined by

$$F(a, b) = \frac{1}{4}(|a| + |b|) + \frac{3}{4}(a^2 + b^2)^{1/2}.$$

It follows from [11, Example 3.1] (see also [10, Propositions 2.1 and 2.2]) that the norm of X is strongly subdifferentiable on S_X . Actually, the duality mapping of X has an easy description. For $(\lambda, y) \in S_X$ with $\lambda \neq 0$ and $y \neq 0$, $D(\lambda, y)$ is singleton, namely, $(\alpha|\lambda|/\lambda, \beta y^*)$, where (α, β) is the unique element of $D(|\lambda|, \|y\|)$ (in $\mathbb{R}^2$ with the norm F) and y^* is the only element in $D(y/\|y\|)$. For $y \in S_Y$, $D(0, y)$ is the set of pairs of the form (α, y^*) , where $|\alpha| \leq \frac{1}{4}$ and y^* is the unique element in $D(y)$. Finally, for $|\lambda| = 1$, we have

$$D(\lambda, 0) = \{(\bar{\lambda}, y^*): y^* \in Y^*, \|y^*\| \leq \frac{1}{4}\}.$$

Thus the diameter of $D(\lambda, y)$ is never greater than $\frac{1}{2}$; hence, X satisfies (5).

Next we show that X fails (3); more concretely, its duality mapping is not upper semicontinuous at $(1, 0)$ for the weak topology of X^* . Since Y is not reflexive, there is a sequence $\{y_n^*\}$ in S_{Y^*} which has no cluster points for the weak topology, and thanks to the Bishop-Phelps Theorem we may arrange that each y_n^* belongs to $D(y_n)$ for some $y_n \in S_Y$. Now let $\{(a_n, b_n)\}$ be a sequence in the unit sphere of $\mathbb{R}^2$ with the norm F such that $0 < b_n < a_n < 1$ for all n , $\{a_n\} \rightarrow 1$, and $\{b_n\} \rightarrow 0$. If (α_n, β_n) is the unique element in $D(a_n, b_n)$, it is easy to check that $\alpha_n < 1$ for all n , $\{\alpha_n\} \rightarrow 1$, and $\{\beta_n\} \rightarrow \frac{1}{4}$. It follows that the sequence $\{(\alpha_n, \beta_n y_n^*)\}$ has no cluster points for the weak topology of X^* ; hence the complement of this sequence is a weakly open subset Ω of X^* . Clearly, $D(1, 0) \subseteq \Omega$, but $D(a_n, b_n y_n)$ is never contained in Ω . Since the sequence $\{(a_n, b_n y_n)\}$ converges in norm to $(1, 0)$, we are done.

3.2. Remarks. (i) It is worth recalling a characterization of quite smooth spaces which shows that they are “smooth” in some sense. If we use D_2 to denote the duality mapping of the bidual X^{**} , then the norm of X is quite smooth at a point $x \in S_X$ if and only if $D_2(x)$ is the w^* -closure in X^{***} of $D(x)$ [11, Theorem 3.1].

(ii) Theorem 1.2 gives us a “roughness” property of any equivalent norm on a non-Asplund space. Using similar arguments, Godefroy and Zizler [15]

have obtained a somehow different kind of “roughness”. The relation between Theorem 1.2 and the results in [15] is not clear to the authors.

(iii) The question if every Asplund space admits an equivalent quite smooth norm was already posed in [11, Problem 3] and remains unanswered. At least we know that being Asplund is a necessary condition.

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