THE REPRESENTATION OF HOMEOMORPHISMS ON THE INTERVAL AS FINITE COMPOSITIONS OF INVOLUTIONS

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ABSTRACT. It is known that every homeomorphism on the interval is the composition of at most four involutions and that every decreasing homeomorphism is the composition of at most three involutions. We will characterize those homeomorphisms that are the composition of two involutions. It is then demonstrated that there exist three involutions that together generate a dense subgroup of the topological group of all homeomorphisms on the interval.

1. INVOLUTIONS

This paper is concerned with the composition of homeomorphisms on the unit interval $[0, 1]$, i.e., strictly increasing or strictly decreasing continuous functions of $[0, 1]$ onto $[0, 1]$. If each of f and g is a homeomorphism, then f is *conjugate* to g if and only if there exists a homeomorphism h such that $f = h^{-1}gh$.

A homeomorphism θ is an *involution* if and only if $\theta(\theta) = \theta^2 = \text{id}$ and $\theta \neq \text{id}$; in other words, θ is a period two homeomorphism. The only periodic homeomorphisms on the interval other than the identity are involutions. The linear involution is denoted by J , $J(x) = 1 - x$.

An increasing $f: [a, b] \rightarrow [a, b]$ is *under the identity* on $[a, b]$ if $f(x) < x$ for all $a < x < b$; f is *above the identity* on $[a, b]$ if $f(x) > x$ for all $a < x < b$.

There is a simple method for constructing an involution f as follows: Choose $0 < a < 1$, and let f_0 be any decreasing homeomorphism of $[0, a]$ onto $[a, 1]$. Then define

$$f(x) = \begin{cases} f_0(x) & \text{for } x \in [0, a], \\ f_0^{-1}(x) & \text{for } x \in [a, 1]. \end{cases}$$

The graph of an involution is characterized by being decreasing and symmetric about the identity graph.

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Another method is to begin with any h that is decreasing on $[0, 1]$; then each of $M(x) = \max\{h(x), h^{-1}(x)\}$ and $m(x) = \min\{h(x), h^{-1}(x)\}$ is an involution.

It is useful to note that any two involutions are conjugate. This can be established by showing that any involution is conjugate to J . Suppose that θ is an involution and $\theta(a) = a$. Let

$$h(x) = \begin{cases} 2ax & \text{for } x \in [0, 1/2], \\ \theta(2a - 2ax) & \text{for } x \in [1/2, 1]. \end{cases}$$

Then $hJ = \theta h$ can be easily checked. Also note that $J = (hJ)^{-1}\theta(hJ)$ is a conjugacy via an orientation reversal. Therefore, any two involutions are conjugate in two ways; by means of an increasing homeomorphism or a decreasing homeomorphism.

2. TWO INVOLUTIONS

What follows is a composite of relevant theorems from [2].

Theorem 2.1. (1) *Every increasing homeomorphism on $[0, 1]$ that is above the identity or below the identity is the composition of two involutions.*

(2) *Every decreasing homeomorphism on $[0, 1]$ is the composition of at most three involutions.*

(3) *Every homeomorphism on $[0, 1]$ is the composition of at most four involutions.*

Thus every increasing homeomorphism is either the composition of two or of four involutions. In this section we will characterize those homeomorphisms that are the composition of two involutions. We begin with a lemma that is only slightly more general than part (1) of Theorem 2.1.

Lemma 2.1. *If f is below the identity or above the identity on $[0, 1]$, $0 < s < 1$, and t is between s and $f(s)$, then there exist involutions θ and ϕ such that $f = \phi\theta$, $\theta(s) = s$, and $\phi(t) = t$.*

Proof. Let us suppose that f is below the identity. For the sake of simplicity of notation we will reduce $f^i(s)$ to s^i for $i \neq 0$ and let $s^0 = s$.

First define $\phi_0: [s^1, s] \rightarrow [s^1, s]$ to be any involution on $[s^1, s]$ that fixes t . Define θ so that $\theta(0) = 1$, $\theta(1) = 0$, and for each i , $\theta: [s^i, s^{i-1}] \rightarrow [s^{-i+1}, s^{-i}]$ by the formula

$$\theta(x) = f^{-i}\phi_0 f^{-i+1}(x).$$

The continuity of θ can be checked by considering the consistency of the definition of θ on the f orbit of s . Note also that $\theta(s) = s$. Now suppose that i is an integer and $x \in [s^i, s^{i-1}]$. Then $\theta(x) \in [s^{-i+1}, s^{-i}]$ and

$$\theta\theta(x) = \theta(f^{-i}\phi_0 f^{-i+1}(x)) = f^{i-1}\phi_0 f^i f^{-i}\phi_0 f^{-i+1}(x) = x.$$

Of course, $\theta\theta(0) = 0$ and $\theta\theta(1) = 1$, and thus θ is an involution.

We now define $\phi = f\theta$ and observe that ϕ is an extension of ϕ_0 since if $i = 1$ and $x \in [s^1, s^0]$, then $\phi(x) = f\theta(x) = ff^{-1}\phi_0 f^0(x) = \phi_0(x)$. Now

suppose that i is any integer and $x \in [s^i, s^{i+1}]$. Then

$$\begin{aligned}\phi\phi(x) &= f\theta f\theta(x) = f\theta(f^{-i+1}\phi_0 f^{-i+2}(x)) \\ &= f^{i-1}\phi_0 f^{i-1} f^{-i+1}\phi_0 f^{-i+1}(x) = x.\end{aligned}$$

And since $\phi^2(0) = 0$ and $\phi^2(1) = 1$, we have that ϕ is an involution and $\phi(t) = t$. Therefore, θ and ϕ are involutions and $\phi\theta = f\theta\theta = f$.

The construction can be summarized by the following diagram:

$$0 < \cdots < s^2 < s^1 \begin{matrix} \xleftarrow{\theta} \\ \xrightarrow{\phi} \end{matrix} s < s^{-1} < s^{-2} < \cdots < 1.$$

The homeomorphism θ imposes a symmetry on the f orbit of s about the point s , while ϕ does the same about the point t that is between s^1 and s . The homeomorphism f serves as a sort of conveyer to move intervals to and from the interval $[s^1, s]$ where ϕ_0 performs the required involution.

If f is above the identity, then the construction is the same with the direction of the orbit reversed in the above diagram and with i replaced by $-i$ in the definition of θ . Better still, notice that if f is above the identity, then JfJ is below the identity, and that if $JfJ = \phi\theta$, then $f = (J\phi J)(J\theta J)$. We note finally that the assignments $\theta(s) = s$ and $\phi(t) = t$ serve to point out that there is a continuum of choices for the involutions θ and ϕ . $\square$

Lemma 2.2. Suppose that $[a, b]$ and $[c, d]$ are subintervals of $[0, 1]$ such that $b \leq c$, $a = 0$ if and only if $d = 1$, and f is an increasing homeomorphism on $[0, 1]$ that fixes each of a, b, c , and d , and is on the same side of the identity on the two intervals $[a, b]$ and $[c, d]$. Then there exist decreasing homeomorphisms θ and ϕ defined on $[a, b] \cup [c, d]$ such that $\theta, \phi: [a, b] \rightarrow [c, d]$, $\theta, \phi: [c, d] \rightarrow [a, b]$, and for all $x \in [a, b] \cup [c, d]$, $f(x) = \phi\theta(x)$ and $\theta^2(x) = \phi^2(x) = x$.

Proof. The proof of this lemma will parallel the proof of Lemma 2.1 to the extent that certain details can be omitted. We begin by choosing numbers s and t , $a < s < b$, $c < t < d$, and consider the f orbit of s and the f orbit of t on the intervals $[a, b]$ and $[c, d]$ respectively. Suppose for now that f is below the identity on both intervals. Let ϕ_0 be any involution of $[s^1, s]$ onto $[t^1, t]$. That is, ϕ_0 is defined on $[s^1, s] \cup [t^1, t]$, $\phi_0: [s^1, s] \rightarrow [t^1, t]$, $\phi_0: [t^1, t] \rightarrow [s^1, s]$, is decreasing on each interval, and $\phi_0^2(x) = x$ for all $x \in [s^1, s] \cup [t^1, t]$. We could make ϕ_0 linear on each interval, for example.

For each integer i and $x \in [s^i, s^{i-1}]$ define $\theta_0(x) = f^{-i}\phi_0 f^{-i+1}(x)$ and thus $\theta_0: [s^i, s^{i-1}] \rightarrow [t^{-i+1}, t^{-i}]$ in a decreasing manner. Also let $\theta_0(a) = d$ and $\theta_0(b) = c$, and the continuity of θ_0 can be easily checked. We now extend θ_0 to a homeomorphism θ defined on $[a, b] \cup [c, d]$ by setting $\theta(x) = \theta_0(x)$ for $x \in [a, b]$ and $\theta(x) = \theta_0^{-1}(x)$ for $x \in [c, d]$. It follows that $\theta^2(x) = x$ for all $x \in [a, b] \cup [c, d]$.

Define $\phi(x) = f\theta(x)$ for all $x \in [a, b] \cup [c, d]$. The homeomorphism ϕ is an extension of ϕ_0 , and we can check that $\phi^2(x) = x$; so ϕ and θ satisfy the conclusion of the lemma. Note that in case $b = c$, $f(b) = \theta(b) = \phi(b) = b$. If f is above the identity on the two intervals, then the direction of the orbits of s and t are reversed and i is replaced by $-i$ in the definition of θ_0 . Finally, we observe that the construction clearly allows for a continuum of choices for θ and ϕ . $\square$

Theorem 2.2. *If f is a homeomorphism on $[0, 1]$, then f is the composition of two involutions if and only if f is increasing and there exists a decreasing homeomorphism ψ on $[0, 1]$ such that for all $x \in [0, 1]$,*

$$\text{sign}(f(x) - x) = \text{sign}(f\psi(x) - \psi(x)).$$

Proof. The closures of the components of $[0, 1]$ minus the fixed point set of f will be referred to as the *complementary intervals*. Geometrically, the condition of the existence of the homeomorphism ψ can be seen as a sort of symmetry on the graph of f in relation to the identity graph about the unique fixed point of ψ . The homeomorphism ψ leaves the set of fixed points of f invariant and matches the complementary intervals in accordance with the two types: *below* the identity, and *above* the identity. See Example 2.1 and Example 2.2.

First, let us suppose that $f = \phi\theta$ with $\phi^2 = \theta^2 = \text{id}$. The required decreasing ψ will be demonstrated to be θ . Since ϕ is decreasing, we have that for any $x \in [0, 1]$, $\text{sign}(f(x) - x) = \text{sign}(\phi\theta(x) - x) = -\text{sign}(\theta(x) - \phi(x)) = \text{sign}(\phi(x) - \theta(x)) = \text{sign}(f\theta(x) - \theta(x))$.

Now suppose that a decreasing homeomorphism ψ exists such that for all $x \in [0, 1]$, $\text{sign}(f(x) - x) = \text{sign}(f\psi(x) - \psi(x))$, and let e be the unique fixed point of ψ . Also suppose that $f \neq \text{id}$ since otherwise the conclusion follows trivially. The strategy for constructing θ and ϕ is to have each of them leave the fixed-point set of f invariant and to operate on the complementary intervals according to the conditions covered by the lemmas. For the purpose of the first objective we will do the following: If x is a fixed point of f and $x \leq e$, then define $\theta(x) = \phi(x) = \psi(x)$; and if x is a fixed point of f with $e \leq x$, then define $\theta(x) = \phi(x) = \psi^{-1}(x)$. The homeomorphisms θ and ϕ are now defined on the fixed point set of f , and clearly, $f(x) = \phi\theta(x)$ and $\phi^2(x) = \theta^2(x) = x$ for all x fixed by f .

We must now define θ and ϕ on the complementary intervals. Consider first the simplifying case where $f(e) = e$. If $[a, b]$ is any complementary interval to the left of e , then it is matched uniquely with the complementary interval $[\psi(b), \psi(a)]$ to the right of e where f is on the same side of the identity on the two intervals. If $a = 0$, then $\psi(a) = 1$; and if $b = e$, then $\psi(b) = e$. We can now apply Lemma 2.2 to define θ and ϕ on these two intervals. This should complete the definition of the homeomorphisms and complete the proof for this case.

Now suppose that $f(e) \neq e$. Then f is either below or above the identity on an interval $[a, b]$ containing e in its interior with $f(a) = a$ and $f(b) = b$. If $a = 0$, then $b = 1$ and Lemma 2.1 finishes the proof. But even if $0 < a < e < b < 1$, we can apply Lemma 2.1 to the interval $[a, b]$ to define θ and ϕ on $[a, b]$. Then the rest of the definition of the two homeomorphisms is done in exactly the same manner as in the previous paragraph. $\square$

Example 2.1. Let f be an increasing homeomorphism on $[0, 1]$ that is under the identity on $[0, \frac{1}{2}]$ and above the identity on $[\frac{1}{2}, 1]$. Then f is not the composition of two involutions.

We describe next a general method for construction of a homeomorphism f that is the composition of two involutions.

Example 2.2. Let g be any increasing homeomorphism on the interval $[0, 1]$, and choose a number $0 < a < 1$. Define

$$f(x) = \begin{cases} ag(x/a) & \text{for } x \in [0, a], \\ 1 - (1-a)g^{-1}((1-x)/(1-a)) & \text{for } x \in [a, 1]. \end{cases}$$

The method, graphically, is to place a copy of g compressed to the interval $[0, a]$ and to place a conjugated copy of g^{-1} compressed to the interval $[a, 1]$. The result has the type of symmetry demanded by Theorem 2.2 even though the set of fixed points of f may be complex enough to contain a Cantor set, for example. One can check that $\text{sign}(f(x) - x) = \text{sign}(f\psi(x) - \psi(x))$ for all $x \in [0, 1]$, where

$$\psi(x) = \begin{cases} 1 - (1-a)x/a & \text{for } x \in [0, a], \\ a(1-x)/(1-a) & \text{for } x \in [a, 1]. \end{cases}$$

3. DENSITY RESULTS

In the construction in Example 2.2, the homeomorphism f could be made arbitrarily close to g by choosing the number a sufficiently close to 1. As a result, we have the following:

Theorem 3.1. *If f is an increasing homeomorphism on $[0, 1]$ and $\epsilon > 0$, then there exist involutions θ and ϕ on $[0, 1]$ such that $|f(x) - \phi\theta(x)| < \epsilon$ for all $x \in [0, 1]$.*

If K is a subset of a semigroup S , then $\Gamma(K)$ denotes the subsemigroup (or subgroup) of S generated by the elements of K , that is, all finite compositions of elements of K . The set of homeomorphisms on $[0, 1]$ under the operation of composition and endowed with the uniform metric is referred to as the *topological group of homeomorphisms on $[0, 1]$* .

We now call upon the following:

Theorem 3.2 [1, Theorem 3.3]. *There exists a homeomorphism ϕ above the identity on $[0, 1]$ such that the semigroup $\Gamma(\phi, J)$ is dense in the topological group of homeomorphisms on $[0, 1]$.*

Applying Lemma 2.1 to express the homeomorphism ϕ from Theorem 3.2 as the composition of two involutions and combining these two factors with J we have the first part of the following:

Theorem 3.3. *There exist involutions θ_1 and θ_2 on $[0, 1]$ such that $\Gamma(\theta_1, \theta_2, J)$ is a dense subgroup of the topological group of homeomorphisms on $[0, 1]$. Furthermore, three is the minimum number of generators of involutions that can generate a dense subset.*

Proof. It should be clear that $\Gamma(\theta_1, \theta_2, J)$ is a group and is dense. For the second part, suppose that θ and ϕ are any two involutions on $[0, 1]$. The increasing elements of $\Gamma(\theta, \phi)$ have an even number of factors and each such element different from the identity reduces to the form $(\theta\phi)^n$ or $(\phi\theta)^n$. And since $\phi\theta = (\theta\phi)^{-1}$, it follows that $\Gamma(\theta, \phi) = \{(\theta\phi)^n : n = 0, \pm 1, \pm 2, \pm 3, \dots\}$. As such, $\Gamma(\theta, \phi)$ clearly is not dense in the topological group of homeomorphisms on $[0, 1]$. There is a continuum of choices for θ_1 and θ_2 . $\square$

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