

KOSZUL COMPLEXES AND HYPERSURFACE SINGULARITIES

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ABSTRACT. The behavior of Poincaré series and Betti numbers of complex projective hypersurfaces in terms of their singularities is compared.

THE PROBLEM

Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a reduced homogeneous polynomial of degree d and

$$f_k = \partial f / \partial x_k, \quad k = 0, 1, \dots, n,$$

its partial derivatives.

Let K be the Koszul (homology) complex of the elements f_0, \dots, f_n in the ring S , where $S = \mathbb{C}[x_0, \dots, x_n]$.

The aim of this note is to consider relations between

- (i) the homology groups $H_k(K)$ of the Koszul complex K , and
- (ii) the singularities of the hypersurface V defined by the equation $f = 0$ in the complex projective space \mathbb{P}^n .

Note that $H_0(K)$ is just the Milnor (or Jacobian) algebra of f given by

$$M(f) = S/(f_0, \dots, f_n).$$

Also note that all the homology groups $H_k(K)$ are graded objects in a natural way (see §1 for details).

For any graded object A we denote by A_m its homogeneous component of degree m and by $P(A)$ the corresponding Poincaré series, i.e.,

$$P(A)(t) = \sum_{m \geq 0} (\dim A_m) t^m.$$

To the best of our knowledge, the only general results relating (i) and (ii) are the following.

Proposition 1. *The following statements are equivalent:*

- (i) $H_k(K) = 0$ for $k > 0$ and

$$P(M(f))(t) = (1 - t^{d-1})^{n+1} / (1 - t)^{n+1}.$$

- (ii) *The hypersurface V is smooth.*

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For a proof see, for instance, [G; D1, p. 109].

Proposition 2. *Let Σ be the singular locus of the hypersurface V . Then*

$$H_k(K) = 0 \text{ for } k > \dim(\Sigma) + 1.$$

For a proof we refer to [Gr, Sa].

In what follows we restrict our attention to the case $\dim(\Sigma) = 0$, i.e., V has only isolated hypersurface singularities, say at the points a_1, \dots, a_p . According to Proposition 2, there are only two nontrivial homology groups in this case, namely, $H_0(K)$ and $H_1(K)$. Moreover, it is easy to show that their Poincaré series satisfy the relation

$$P(H_0(K))(t) - P(H_1(K))(t) = (1 - t^{d-1})^{n+1} / (1 - t)^{n+1}.$$

Hence we are left with the *problem* of computing just one of them, say $P(H_0(K))$ or $P(M(f))$.

It is known that there are relations between the homology groups $H_k(K)$ and the topology of the hypersurface V (for the smooth case refer to [G] and for the singular case to [D3]). These relations, expressed in general by an intricate spectral sequence [D3], turn out to be more qualitative than quantitative.

Our results can be stated loosely but, we hope, vividly as follows.

Theorem 3. (A) *The Euler characteristic $\chi(V)$ and the dimensions $\dim M(f)_m$ for $m > (n + 1)(d - 2)$ depend only on local invariants of the singularities (V, a_j) , for example, their Milnor numbers $\mu(V, a_j)$ and their Tjurina numbers $\tau(V, a_j)$.*

(B) *The Betti numbers $b_m(V)$ and the Poincaré series $P(M(f))$ depend in general not only on local invariants but also on the position of the singularities of the hypersurface V .*

(C) *Let $V: f = 0$ and $\bar{V}: \bar{f} = 0$ be two hypersurfaces as above. Consider the statements*

(α) $P(M(f)) = P(M(\bar{f}))$.

(β) *The pairs (\mathbb{P}^n, V) and (\mathbb{P}^n, \bar{V}) are homeomorphic.*

Then neither $(\alpha) \Rightarrow (\beta)$ nor $(\beta) \Rightarrow (\alpha)$ holds in general.

Note that the topological part in (A) and (B) is known due to the classical examples by Zariski in 1929 (for more details and other references on this part see [D2] or [D4]). The proof for the statements involving $M(f)$ can be found below, together with additional facts and interesting examples

1. HOMOGENEOUS KOSZUL COMPLEXES

Our general reference for Koszul complexes is Matsumura [M, pp. 132–136]. We need to pay more attention to the homogeneity properties of our complexes. Consider the polynomial ring $S = \mathbb{C}[x_0, \dots, x_n]$ with its usual grading. For any homogeneous polynomial $g \in S_{|g|}$, $|g| = \deg(g)$, we define a complex

$$K(g): 0 \rightarrow Se_g \xrightarrow{d} S \rightarrow 0$$

where $d(e_g) = g$ and d is S -linear. Here we regard e_g as an homogeneous vector of degree $|e_g| = \deg(g)$. Then d is a homogeneous morphism of degree

0; i.e., it preserves the homogeneous components if we set $|he_g| = |h| + |g|$. For s homogeneous polynomials g_1, \dots, g_s the tensor product

$$K(g_1, \dots, g_s) = K(g_1) \otimes \dots \otimes K(g_s)$$

is exactly the Koszul complex of the elements g_1, \dots, g_s in S . Since the corresponding differential d has degree 0, it follows that all the homology groups $H_k(K(g_1, \dots, g_s))$ are graded \mathbb{C} -vector spaces.

Let g, g_1, \dots, g_s be as above. Consider the exact sequence of complexes

$$0 \rightarrow S \rightarrow K(g) \rightarrow Se_g \rightarrow 0$$

(here Se_g is the complex $0 \rightarrow Se_g \rightarrow 0 \rightarrow 0$, while S denotes the complex $0 \rightarrow 0 \rightarrow S \rightarrow 0$). Tensorizing by $K(g_1, \dots, g_s)$ we get an exact sequence of complexes

$$0 \rightarrow K(g_1, \dots, g_s) \rightarrow K(g, g_1, \dots, g_s) \rightarrow \widehat{K} \rightarrow 0$$

where $(\widehat{K}_m)_q = (K(g_1, \dots, g_s)_{m-1})_{q-|g|}$. Here q and $q - |g|$ indicate homogeneity components. Using a lemma on p. 135 in [M] we get

Lemma 4. *For any positive integer q there is a long exact sequence*

$$\begin{aligned} \rightarrow H_m(K(g_1, \dots, g_s))_{q-|g|} \xrightarrow{g} H_m(K(g_1, \dots, g_s))_q \\ \rightarrow H_m(K(g, g_1, \dots, g_s))_q \xrightarrow{\delta} H_{m-1}(K(g_1, \dots, g_s))_{q-|g|} \rightarrow \dots \end{aligned}$$

where g denotes the morphism induced by the multiplication by g .

Let us come back now to the hypersurface V . Since V has only isolated singularities by assumption, we can find a hyperplane H in \mathbb{P}^n such that the intersection $V \cap H$ is smooth. We can even assume that $x_0 = 0$ is an equation for H . It follows that x_0, f_1, \dots, f_n is a regular sequence in S ; the same holds for the subsequence f_1, \dots, f_n .

We apply Lemma 4 taking $g = x_0$ and $g_j = f_j$ for $j = 0, \dots, n$. To simplify notation, we set

$$\widetilde{K} = K(x_0, f_0, \dots, f_n) \quad \text{and} \quad \overline{K} = K(x_0, f_1, \dots, f_n).$$

Then we get the exact sequence

$$(5) \quad H_1(\widetilde{K})_q \rightarrow M(f)_{q-1} \xrightarrow{x_0} M(f)_q \rightarrow H_0(\widetilde{K})_q \rightarrow 0.$$

Using again Lemma 4 with $g = f_0$, $g_0 = x_0$, and $g_j = f_j$ for $j = 1, \dots, n$, we get the exact sequence

$$(6) \quad 0 \rightarrow H_1(\widetilde{K})_q \rightarrow H_0(\overline{K})_{q-d+1} \xrightarrow{f_0} H_0(\overline{K})_q \rightarrow H_0(\widetilde{K})_q \rightarrow 0.$$

Since x_0, f_1, \dots, f_n is a regular sequence, it follows that

$$P(H_0(\overline{K}))(t) = (1 - t^{d-1})^n / (1 - t)^n$$

(see, for instance, [D1, p. 109]). In particular,

$$(7) \quad \begin{aligned} H_0(\widetilde{K})_q &= 0 \quad \text{for } q > n(d-2), \\ H_0(\widetilde{K})_q &= 0 \quad \text{for } q > (n+1)(d-2) + 1, \end{aligned}$$

$$\dim H_1(\widetilde{K})_{(n+1)(d-2)+1} = 1.$$

Using (7) and (5) we finally get

Corollary 8. *The morphism $x_0: M(f)_{q-1} \rightarrow M(f)_q$ is an epimorphism for $q > n(d-2)$ and an isomorphism for $q > (n+1)(d-2) + 1$. Moreover,*

$$\dim \ker\{x_0: M(f)_{(n+1)(d-2)} \rightarrow M(f)_{(n+1)(d-2)+1}\} \leq 1.$$

Corollary 9. *$\dim M(f)_q = \tau(V)$ for all $q > (n+1)(d-2)$, where $\tau(V)$ is the sum of all Tjurina numbers $\tau(V, a_j)$.*

$\tau(V)$ is sometimes called the total Tjurina number of the hypersurface V .

Proof. It follows from Corollary 8 that the dimensions $\dim M(f)_q$ are constant for $q > (n+1)(d-2)$.

Since the ideal (f_0, \dots, f_n) defines a 0-dimensional scheme Y whose support is $\{a_1, \dots, a_p\}$, for large enough q we have

$$\dim M(f)_q = \sum_{i=1, p} \dim(M(f))_{a_i}$$

where $(\)_{a_i}$ means localization at the point a_i . To see this, use [H, Exercise 5.2, p. 230] with $X = \mathbb{P}^n$ and \mathcal{F} the structure sheaf of the scheme Y . Assume that $a_1 = (1: 0: \dots: 0)$, and set $y_j = x_j/x_0$ for $j = 1, \dots, n$. Using a mild GAGA type result, one can see that

$$M(f)_{a_1} \simeq \mathbb{C}\{y_1, \dots, y_n\}/(h_0, \dots, h_n)$$

where $h_j(y) = f_j(1, y)$ and $\mathbb{C}\{y_1, \dots, y_n\}$ denotes the convergent power series in y_1, \dots, y_n . The singularity (V, a_1) is locally defined by the equation $h(y) = f(1, y) = 0$. Note that

$$\partial h / \partial y_j = h_j \quad \text{for } j > 0$$

and

$$dh(y) = h_0(y) + y_1 h_1(y) + \dots + y_n h_n(y)$$

(use the Euler relation!).

Hence the ideals (h_0, \dots, h_n) and $(h, \partial h / \partial y_1, \dots, \partial h / \partial y_n)$ are equal.

It follows that

$$\dim M(f)_{a_1} = \dim \frac{\mathbb{C}\{y_1, \dots, y_n\}}{(h, \partial h / \partial y_1, \dots, \partial h / \partial y_n)} = \tau(V, a_1)$$

according to the definition of the Tjurina numbers (see, for instance, [D1, p. 90]). This ends the proof of Corollary 9 and Theorem 3(A).

(A different, more geometric, proof of this result is contained in [D3, Theorem (3.9)].) A remark is in order about the other nontrivial homology group $H_1(K)$. Its elements can be identified to nontrivial relations (syzygies)

$$(10) \quad b_0 f_0 + \dots + b_n f_n = 0.$$

It is easy to see, using sequences similar to (5) and (6) that one has

Corollary 11. *The morphism $x_0: H_1(K)_{q-1} \rightarrow H_1(K)_q$ is injective for all q and an isomorphism for all $q > (n+1)(d-2) + 1$. Moreover,*

$$\dim \text{coker}\{x_0: H_1(K)_{(n+1)(d-2)} \rightarrow H_1(K)_{(n+1)(d-2)+1}\} \leq 1.$$

In other words, there are exactly $\tau(V)$ nontrivial relations (10) with $\deg(b_k) = n(d-2)$.

Any relation (10) with $\deg(b_k) > n(d - 2)$ is obtained from the previous ones by multiplying by a suitable power of x_0 . And there are at least $\tau(V) - 1$ nontrivial relations (10) with $\deg(b_k) = n(d - 2) - 1$. Therefore, the presence of singularities on the hypersurface V forces some nontrivial relations for the partial derivatives f_0, \dots, f_n !

2. EXAMPLES OF POINCARÉ SERIES $P(M(f))$

The simplest example of a Poincaré series $P(M(f))$ is when this series is a polynomial. This happens exactly when the hypersurface V is *smooth*; the corresponding formula for $P(M(f))$ is given in Proposition 1.

Example 12 (smooth quartic curves in \mathbb{P}^2). Take $n = 2$ and $d = 4$. Then the formula in Proposition 1 implies

$$P(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + t^6.$$

Beyond the smooth case, there is just one general situation in which we know explicit formulas for $P(M(f))$.

The treatment below is based on results by Pellikaan [P] and van Straten [vS] and, in a special case (transversal type A_1), was already discussed by Siersma [S]. Let $J_f = (f_0, \dots, f_n)$ be the Jacobian ideal of our polynomial f in S .

Assume that there are homogeneous polynomials g_1, \dots, g_n in S such that

$$J_f \subset I \quad \text{and} \quad (J_f)_s = I_s \quad \text{for all } s \gg 0$$

where I is the ideal generated by g_1, \dots, g_n in S . In geometric terms, this means that the scheme Y defined by J_f in \mathbb{P}^n is a complete intersection. Hint: use [H, 5.9 and 5.10, p. 125] to show that the scheme defined by the ideal I is also Y ! Now write

$$df = \sum_{i=1}^n g_i w_i$$

for some 1-forms w_i on \mathbb{C}^{n+1} . The coefficients of w_i are homogeneous polynomials of degree $d - a_i - 1$, where $a_i = \deg(g_i)$ for $i = 1, \dots, n$.

Then Pellikaan and van Straten have shown that the relation R in $H_1(K)$ corresponding to the obvious relation

$$df \wedge w_1 \wedge \dots \wedge w_n = 0$$

generates $H_1(K)$ as a free S/I -module. Since g_1, \dots, g_n define a complete intersection, it follows that

$$P(S/I)(t) = \frac{(1 - t^{a_1}) \dots (1 - t^{a_n})}{(1 - t)^{n+1}}$$

(see, for instance, [D1, p. 109]).

Since $\deg(R) = (n + 1)(d - 1) - \sum a_i$, we get

Proposition 13. *Assume that the singular locus Y of the hypersurface V is a 0-dimensional complete intersection of type (a_1, \dots, a_n) . Then*

$$P(M(f))(t) = \frac{(1 - t^{d-1})^{n+1}}{(1 - t)^{n+1}} + t^{(n+1)(d-1) - \sum a_i} \frac{(1 - t^{a_1}) \dots (1 - t^{a_n})}{(1 - t)^{n+1}}.$$

Example 14. (i) $\dim V = 0$ ($n = 2$). Any polynomial f in two variables x_0 and x_1 can be written as

$$f = (\alpha_1 x_0 + \beta_1 x_1)^{k_1} \cdots (\alpha_m x_0 + \beta_m x_1)^{k_m}$$

where $k_1 + \cdots + k_m = d$ and the points $(\beta_i : -\alpha_i)$ are distinct in \mathbb{P}^1 . Such a polynomial f is not reduced as soon as some k_j is larger than one, but the corresponding hypersurface V has only isolated singularities (since it consists only of finitely many points!). Therefore, we can apply our results above to this polynomial f .

It is easy to see that

$$\gcd(f_0, f_1) = g_1$$

where

$$g_1 = (\alpha_1 x_0 + \beta_1 x_1)^{k_1-1} \cdots (\alpha_m x_0 + \beta_m x_1)^{k_m-1}.$$

It follows that Y is a complete intersection (defined by $g_1 = 0$) of type $a_1 = d - m$. Hence, using Proposition 13, we get

$$P(M(f))(t) = \frac{1 - 2t^{d-1} + t^{d+m-2}}{(1-t)^2}$$

where m is the number of distinct factors in f .

(ii) V is a union of two smooth curves C_1 and C_2 in \mathbb{P}^2 , meeting transversally. Let $C_1: g_1 = 0$ and $C_2: g_2 = 0$ be equations for the smooth curves C_1 and C_2 . If $V = C_1 \cup C_2$ has only nodes as singularities (coming from the points in $C_1 \cap C_2$), then the scheme Y is a complete intersection defined by $g_1 = g_2 = 0$.

Let $f = g_1 g_2$, $\alpha_1 = \deg(g_1)$, $a_2 = \deg(g_2)$, and $d = \deg(f) = a_1 + a_2$. Using Proposition 9, we get

$$P(M(f))(t) = \frac{(1 - t^{d-1})^3 + t^{2d-3}(1 - t^{a_1})(1 - t^{a_2})}{(1-t)^3}.$$

As a numerical example, consider the case $a_1 = 1$ and $a_2 = 3$; i.e., V is a quartic curve consisting of a line and a general cubic curve. Then

$$P(M(f)) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 4t^5 + 3t^6 + 3t^7 + \cdots.$$

As a concrete example, one can take $V: x_0(x_0^3 + x_1^3 + x_2^3) = 0$.

Remark 15. Consider the quartic curve C defined by

$$f = x_0^2 x_1^2 + x_1^2 x_2^2 + x_2^2 x_0^2.$$

It is easy to show that C has as singularities three nodes A_1 at the points $(1: 0: 0)$, $(0: 1: 0)$, and $(0: 0: 1)$.

Moreover, a direct computation shows that there are no relations (10) with $\deg(b_k) = 2$ in this case.

Hence $\dim M(f)_5 = 3$, and using also Corollary 8 we get the Poincaré series

$$P(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + 3t^6 + \cdots.$$

In particular, this quartic C and the quartic V from the end of Example 14 have distinct Poincaré series, in spite of the fact that they both have three nodes A_1 as singularities.

This proves Theorem 3(B). For readers who may want such examples involving only *irreducible* hypersurfaces, let us consider the surfaces in \mathbb{P}^3 given by

$$S_1: x_0(x_0^3 + x_1^3 + x_2^3) + x_3^4 = 0,$$

$$S_2: x_0^2x_1^2 + x_1^2x_2^2 + x_2^2x_0^2 + x_3^4 = 0.$$

Then S_1 and S_2 are both irreducible surfaces and have three singularities of type A_3 as their singular loci and distinct Poincaré series.

To see this last fact, assume that $f \in \mathbb{C}[x_0, \dots, x_n]$ and $g \in \mathbb{C}[y_0, \dots, y_m]$ are both homogeneous polynomials of degree d . Then it is easy to see that their Poincaré series satisfy a “Thom-Sebastiani type” result, namely,

$$(16) \quad P(M(f + g)) = P(M(f)) \cdot P(M(g))$$

where $f + g \in \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_m]$.

In fact, using this formula one can compute many other examples starting from the examples given in this note.

Example 17. In this example we list the Poincaré series for the *plane cubic curves*. The results follow by simple direct computations with the normal forms for such cubic curves V (see, for instance, [D1, p. 51]). The funny thing about these Poincaré series is that they depend *only* on the total Tjurina number $\tau(V)$. With obvious notation, we have

$$\begin{aligned} \tau(V) = 0: P(\text{smooth})(t) &= 1 + 3t + 3t^2 + t^3, \\ \tau(V) = 1: P(\text{nodal})(t) &= P(\text{smooth})(t) + t^4(1 - t)^{-1}, \\ \tau(V) = 2: P(\text{cuspidal})(t) &= P(\text{conic} + \text{chord})(t) \\ &= P(\text{smooth})(t) + t^3(1 + t)(1 - t)^{-1}, \\ \tau(V) = 3: P(\text{conic} + \text{tangent})(t) &= P(\text{triangle})(t) \\ &= P(\text{smooth})(t) + t^3(2 + t)(1 - t)^{-1}. \end{aligned}$$

In particular, they show that the implication $(\alpha) \Rightarrow (\beta)$ in Theorem 3(C) does not hold in general.

Example 18. Consider the family of plane curves in \mathbb{P}^2 given by

$$C_\lambda: f_\lambda = x_2(x_0^5 + x_1^5) + \lambda x_0^3x_1^3 + x_0^6 + x_1^6$$

for λ in a neighbourhood of $0 \in \mathbb{C}$. Any curve C_λ has just one singular point, namely, the point $a_1 = (0 : 0 : 1)$. It is also easy to see that

$$\mu(C_\lambda, a_1) = 16,$$

so we have a μ -constant family. Such families of plane curves are equisingular [T]; in particular, the topological type of the pair $(\mathbb{P}^2, C_\lambda)$ (up to homeomorphism) does not depend on λ .

On the other hand, $\tau(C_0, a_1) = 16$ while $\tau(C_\lambda, a_1) = 15$ for $\lambda \neq 0$ (similar computations can be found, for instance, in [D1, pp. 94–97]).

As a result, using Corollary 9, it follows that the Poincaré series $P(M(f_\lambda))$ for $\lambda = 0$ and for $\lambda \neq 0$ are distinct.

This shows that the implication $(\beta) \Rightarrow (\alpha)$ from Theorem 3(C) does not hold in general.

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