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NONEXISTENCE OF WEAKLY ALMOST COMPLEX STRUCTURES ON GRASSMANNIANS

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ABSTRACT. In this paper we prove that, for $2 \le k \le n/2$, the unoriented Grassmann manifold $G_k(\mathbb{R}^n)$ admits a weakly almost complex structure if and only if $n = 2k = 4$ or 6; for $3 \le k \le \frac{n}{2}$, none of the oriented Grassmann manifolds $\widetilde{G}_k(\mathbb{R}^n)$—except $\widetilde{G}_3(\mathbb{R}^6)$ and a few as yet undecided ones—admits a weakly almost complex structure.

1. INTRODUCTION

For $1 \le k < n$, let $\widetilde{G}_k(\mathbb{R}^n)$ ($G_k(\mathbb{R}^n)$ resp.) denote the oriented (unoriented) Grassmann manifold of the oriented (unoriented) k-dimensional vector subspace of $\mathbb{R}^n$. $\widetilde{G}_k(\mathbb{R}^n)$ ($G_k(\mathbb{R}^n)$) is a smooth manifold of dimension $k(n-k)$. Note that $\widetilde{G}_1(\mathbb{R}^n) \cong S^{n-1}(G_1(\mathbb{R}^n) \cong RP^{n-1})$, the $(n-1)$-sphere (real projective space), and that $\widetilde{G}_k(\mathbb{R}^n) \cong \widetilde{G_{n-k}}(\mathbb{R}^n)$ $(G_k(\mathbb{R}^n) \cong G_{n-k}(\mathbb{R}^n))$ under the diffeomorphism that sends a k-plane V to its orthogonal complement $V^\perp$.

Recall that a smooth manifold M is said to be (weakly) almost complex if its tangent bundle τM is (stably) isomorphic to the realification of a complex vector bundle over M.

For example, $\widetilde{G}_1(\mathbb{R}^n) \cong S^{n-1}$ is weakly almost complex for all n but is almost complex only when $n = 3$ or 7 [1]; $G_1(\mathbb{R}^n) \cong RP^{n-1}$ is weakly almost complex only when n is even. It is a classical result that $\widetilde{G}_2(\mathbb{R}^n) \cong \mathrm{SO}(n)/(\mathrm{SO}(2) \times \mathrm{SO}(n-2))$ is a Hermitian symmetric space and is therefore almost complex for all n. Our main results are:

Theorem 1.1. *Let* $2 \le k \le \frac{n}{2}$. *Then* $G_k(\mathbb{R}^n)$ *is weakly almost complex if and only if* $n = 2k = 4$ *or* 6.

Theorem 1.2. *Let* $3 \le k \le \frac{n}{2}$. *Then* $\widetilde{G}_k(\mathbb{R}^n)$ *is not weakly almost complex if* n *is odd or* $(n-k) \ge 8$.

Our results are sharper than that in [6]. Note that $\widetilde{G}_3(\mathbb{R}^6)$ is weakly almost complex [6]. The unsolved cases for weak complexity of $\widetilde{G}_k(\mathbb{R}^n)$ are $\widetilde{G}_4(\mathbb{R}^8)$, $\widetilde{G}_5(\mathbb{R}^{10})$, $\widetilde{G}_6(\mathbb{R}^{12})$, $\widetilde{G}_7(\mathbb{R}^{14})$, $\widetilde{G}_3(\mathbb{R}^8)$, $\widetilde{G}_4(\mathbb{R}^{10})$, $\widetilde{G}_5(\mathbb{R}^{12})$, and $\widetilde{G}_3(\mathbb{R}^{10})$. Let

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$\widetilde{\gamma_{n,k}}(\gamma_{n,k})$ denote the canonical k-plane bundle over $\widetilde{G}_k(\mathbb{R}^n)$ $(G_k(\mathbb{R}^n))$, and let $\widetilde{\beta_{n,k}}$ $(\beta_{n,k})$ be its orthogonal complement, whose fiber over a $V \in \widetilde{G}_k(\mathbb{R}^n)$ $(G_k(\mathbb{R}^n))$ is the vector space $V^{\perp} \subset \mathbb{R}^n$. We have bundle equivalence

$$\widetilde{\gamma_{n,k}} \oplus \widetilde{\beta_{n,k}} \cong n\varepsilon \qquad (\gamma_{n,k} \oplus \beta_{n,k} \cong n\varepsilon), \tag{1.3}$$

where ε denotes a trivial line bundle.

It is well known that the tangent bundle $\tau\widetilde{G}_k(\mathbb{R}^n)$ $(\tau G_k(\mathbb{R}_n))$ of $\widetilde{G}_k(\mathbb{R}^n))$ has the following description (see [4]):

$$\tau\widetilde{G}_k(\mathbb{R}^n) \cong \widetilde{\gamma_{n,k}} \otimes \widetilde{\beta_{n,k}} \qquad (\tau G_k(\mathbb{R}^n) \cong \gamma_{n,k} \otimes \beta_{n,k}). \tag{1.4}$$

Using (1.3) and (1.4), we obtain

$$\tau\widetilde{G}_k(\mathbb{R}^n) \oplus (\widetilde{\gamma_{n,k}} \otimes \widetilde{\gamma_{n,k}}) \cong n\widetilde{\gamma_{n,k}} \qquad (\tau G_k(\mathbb{R}^n) \oplus (\gamma_{n,k} \otimes \gamma_{n,k}) \cong n\gamma_{n,k}). \tag{1.5}$$

For a CW complex X, let $r\colon K(X) \to KO(X)$ denote the homomorphism of Abelian groups gotten by restriction of scalars to $\mathbb{R}$, and let $c\colon KO(X) \to K(X)$ denote the complexification, $c[\xi] = [\xi \otimes_{\mathbb{R}} \mathbb{C}]$, which is a ring homomorphism.

We have the following identity:

$$rc(x) = 2x \quad \forall x \in KO(X). \tag{1.6}$$

2. The unoriented Grassmannians

Lemma 2.1. *$G_2(\mathbb{R}^6)$ is not weakly almost complex.*

Proof. It is well known that

$$H^*(G_2(\mathbb{R}^6); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \overline{w}_1, \overline{w}_2, \overline{w}_3, \overline{w}_4]$$

modulo the relation $(1 + w_1 + w_2)(1 + \overline{w}_1 + \overline{w}_2 + \overline{w}_3 + \overline{w}_4) = 1$, so

$$H^*(G_2(\mathbb{R}^6); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2]/\langle w_1^5 + w_1w_2^2, w_1^2w_2^2 + w_1^4w_2 + w_2^3\rangle.$$

The fact $H^8(G_2(\mathbb{R}^6); \mathbb{Z}_2) \cong \mathbb{Z}_2$ implies $w_2^4 \neq 0$. By (1.5), the total Stiefel-Whitney classes of $G_2(\mathbb{R}^6)$ are given by

$$w(G_2(\mathbb{R}^6)) = (1 + w_1 + w_2)^6/(1 + w_1^2) = 1 + (w_1^4 + w_2^2) + w_1^2w_2^2 + w_2^4.$$

This gives

$$w_2(G_2(\mathbb{R}^6)) = 0, \qquad w_8(G_2(\mathbb{R}^6)) = w_2^4 \neq 0.$$

The following results follow immediately from Wu's formula $sq^1w_2 = w_1w_2$ [5]:

$$sq(w_1^6) = w_1^6, \qquad sq(w_1^4w_2) = w_1^4w_2 + w_1^5w_2,$$
$$sq(w_1^2w_2^2) = w_1^2w_2^2, \qquad sq(w_2^3) = w_2^3 + w_2^3w_1.$$

Therefore, $sq^2\colon H^6(G_2(\mathbb{R}^6); \mathbb{Z}_2) \to H^8(G_2(\mathbb{R}^6); \mathbb{Z}_2)$ is zero. Hence, $w_8(G_2(\mathbb{R}^6))$ is not in the image of $H^6(G_2(\mathbb{R}^6); \mathbb{Z})$ under the homomorphism sq^2. Our lemma immediately follows from the following criterion [3]: M^8 admits a weakly almost complex structure iff $\delta w_2(M) = 0$ and $w_8(M) \in sq^a 2H^6(M; \mathbb{Z})$.

Lemma 2.2. *If $G_k(\mathbb{R}^n)$ is weakly almost complex, then so are $G_{k-1}(\mathbb{R}^{n-2})$ and $G_k(\mathbb{R}^{n-2})$.*

Proof. Let us consider the maps

$$G_{k-1}(\mathbb{R}^{n-2}) \xrightarrow{i} G_{k-1}(\mathbb{R}^{n-1}) \xrightarrow{j} G_k(\mathbb{R}^n)$$

where i regards a V in $\mathbb{R}^{n-2}$ as a V in $\mathbb{R}^{n-1}$ and j sends a V to $V \oplus \mathbb{R}$. It is easy to see that

$$\begin{aligned} i^*(\gamma_{n-1,k-1}) &\cong \gamma_{n-2,k-1}, & i^*(\beta_{n-1,k-1}) &\cong \beta_{n-2,k-1} \oplus \varepsilon \\ j^*(\gamma_{n,k}) &\cong \gamma_{n-1,k-1} \oplus \varepsilon, & j^*(\beta_{n,k}) &\cong \beta_{n-1,k-1}. \end{aligned} \tag{2.3}$$

So we have

$$\begin{aligned} (j \circ i)^* \tau G_k(\mathbb{R}^n) &\cong i^* \circ j^*(\gamma_{n,k} \otimes \beta_{n,k}) \\ &\cong i^*(\gamma_{n-1,k-1} \oplus \varepsilon) \otimes i^*(\beta_{n-1,k-1}) \\ &\cong (\gamma_{n-2,k-1} \oplus \varepsilon) \otimes (\beta_{n-2,k-1} \oplus \varepsilon) \\ &\cong \gamma_{n-2,k-1} \otimes \beta_{n-2,k-1} \oplus \gamma_{n-2,k-1} \oplus \beta_{n-2,k-1} \oplus \varepsilon \\ &\cong \tau G_{k-1}(\mathbb{R}^{n-2}) \oplus (n-1)\varepsilon. \end{aligned}$$

So the conclusion for $G_{k-1}(\mathbb{R}^{n-2})$ is true.

Let us consider the maps

$$G_k(\mathbb{R}^{n-2}) \xrightarrow{i_1} G_k(\mathbb{R}^{n-1}) \xrightarrow{i_2} G_k(\mathbb{R}^n).$$

By (2.3), we obtain

$$\begin{aligned} (i_2 \circ i_1)^* \tau G_k(\mathbb{R}^n) &\cong i_1^* \circ i_2^*(\gamma_{n,k} \otimes \beta_{n,k}) \cong i_1^*(\gamma_{n-1,k}) \otimes (i_1^*(\beta_{n-1,k}) \oplus \varepsilon) \\ &\cong \gamma_{n-2,k} \otimes (\beta_{n-2,k} \oplus \varepsilon \oplus \varepsilon) \cong \tau G_k(\mathbb{R}^{n-2}) \oplus 2\gamma_{n-2,k}. \end{aligned}$$

By (1.6), $2\gamma_{n-2,k}$ is in the image of $r\colon K(G_k(\mathbb{R}^{n-2})) \to KO(G_k(\mathbb{R}^{n-2}))$. This completes the proof.

Proof of Theorem 1.1. The statement that $G_2(\mathbb{R}^4)$ and $G_3(\mathbb{R}^6)$ are weakly almost complex was obtained in [6].

We note that $G_2(\mathbb{R}^{2n+1})$ is not weakly almost complex, since it is not orientable. The "only if" part of the theorem may be shown by using this fact, Lemma 2.1, and Lemma 2.2 repeatedly.

Remark. Borel and Hirzebruch [2, p. 526] proved that $G_2(\mathbb{R}^n)$ is not almost complex if $n \geq 5$. We extend their results.

3. The oriented Grassmannians

Proof of Theorem 1.2. If n is odd, $3 \leq k \leq n/2$, then $\widetilde{G}_k(\mathbb{R}^n)$ is not weakly almost complex. The reason is that $w_3(\widetilde{G}_k(\mathbb{R}^n)) \neq 0$ [6].

By Lemma 2.1, $G_2(\mathbb{R}^6)$ is not weakly almost complex. But $\tau G_2(\mathbb{R}^6) \oplus (\gamma_{6,2} \otimes \gamma_{6,2}) \cong 6\gamma_{6,2}$. So we see that the element $\gamma_{6,2} \otimes \gamma_{6,2}$ is not in the image of $r\colon K(G_2(\mathbb{R}^6)) \to KO(G_2(\mathbb{R}^6))$.

Let ξ denote the line bundle whose $w_1(\xi)$ equals $w_1(\gamma_{6,2})$. Then $\xi \oplus \gamma_{6,2}$ is an orientable 3-plane bundle with

$$(\xi \oplus \gamma_{6,2}) \otimes (\xi \oplus \gamma_{6,2}) \cong \gamma_{6,2} \otimes \gamma_{6,2} \oplus 2\gamma_{6,2} \otimes \xi \oplus \varepsilon.$$

Then we have

$$(\xi \oplus \gamma_{6,2})^2 \oplus \varepsilon \notin \operatorname{Im} r. \tag{3.1}$$

Now let n be even, $k \geq 3$, and $n-k \geq 8 = \dim G_2(\mathbb{R}^6)$. Since $\widetilde{G}_k(\mathbb{R}^n)$ is $(n-k)$-universal for orientable k-plane bundles, there exists a map $f\colon G_2(\mathbb{R}^6) \to$

$\widetilde{G}_k(\mathbb{R}^n)$ such that $f^*(\tilde{\gamma}_{n,k}) \cong \xi \oplus \gamma_{6,2} \oplus m\varepsilon$, where $m = k - 3$. We have

$$f^*(\tilde{\gamma}_{n,k} \otimes \tilde{\gamma}_{n,k}) \cong (\xi \oplus \gamma_{6,2})^2 \oplus m^2\varepsilon \oplus 2m(\xi \oplus \gamma_{6,2}),$$
$$f^*\tau\widetilde{G}_k(\mathbb{R}^n) \oplus (\xi \oplus \gamma_{6,2})^2 \oplus m^2\varepsilon \oplus 2m(\xi \oplus \gamma_{6,2}) \cong nf^*(\tilde{\gamma}_{n,k}).$$

Using (3.1), (1.6), and the fact that n is even, we see that $\widetilde{G}_k(\mathbb{R}^n)$ is not weakly almost complex. This completes the proof of the theorem.

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