PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 121, Number 4, August 1994

KÄHLER-EINSTEIN SURFACES WITH NONPOSITIVE BISECTIONAL CURVATURE

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(Communicated by Peter Li)

ABSTRACT. In this note we show that, for a Kähler-Einstein surface M with negative Ricci curvature and nonpositive bisectional curvature, if the cotangent bundle of M is not quasi-ample then M is a quotient of the bidisc.

1. INTRODUCTION AND STATEMENT OF RESULT

Let (M, g) be a compact Kähler-Einstein surface with nonpositive bisectional curvature. Assume $c_1 < 0$ (otherwise (M, g) is flat). Then the following was raised in [SY].

Conjecture. Under the above assumptions, (M, g) is isometric to a locally hermitian symmetric surface, namely, a quotient of the ball B^2 or the bidisc $D \times D$.

For the ball quotient case, the first partial answer was obtained by Siu and Yang [SY] in 1981. We need some notation to describe their theorem.

Denote by $H_{\min}$, H_{av}, and $H_{\max}$ the minimum, average, and maximum values of the holomorphic sectional curvature in all directions at p (note that $H_{\text{av}} = \frac{1}{3}$ scalar curvature is a constant). Let $a(p) = H_{\text{av}} - H_{\min}$ and $b(p) = H_{\max} - H_{\text{av}}$. Then for any K-E metric one always has $\frac{1}{2}b(p) \leq a(p) \leq 2b(p)$. Let $\lambda_0 = 2/(1 + 3\sqrt{\frac{6}{11}})$ (≈ 0.622). Their theorem is

Theorem (Siu-Yang). *Let (M, g) be a compact Kähler-Einstein surface with nonpositive bisectional curvature and $c_1 < 0$. If, for any $p \in M$, $a(p) \leq \lambda b(p)$ for some $\lambda < \lambda_0$, then M is a ball quotient.*

In [P] Polombo improved this result by enlarging λ_0 to $\frac{48}{52} \approx 0.923$. (The results in [P] are actually more general.)

Note that for the bidisc case $a(p) \equiv 2b(p) > 0$ identically, so it is the other end of the story.

Received by the editors November 11, 1992.
1991 *Mathematics Subject Classification*. Primary 53C55.
Key words and phrases. Kähler-Einstein, bisectional curvature, quasi-ample.
Research supported by National Science Foundation Grant DMS-91-05185 and Duke University.

For a compact Kähler manifold M, let $\pi: P = \mathbf{P}(T_M) \to M$ be the projectivized tangent bundle and L the tautological line bundle (such that $\pi_*(L) = \Omega_M$, the cotangent bundle). Then by definition Ω_M is ample (nef) if the line bundle L is ample. Closely related to this is the following:

Definition. Ω_M is said to be *quasi-ample* if L is nef and $Y \cdot L^{\dim Y} > 0$ for any irreducible subvariety $Y \subseteq P$ with $\pi(Y) = M$.

Note that there are many examples of a compact Kähler manifold (M, g) with nonpositive bisectional curvature such that $c_1(M) < 0$ but Ω_M is not quasi-ample. In the surface case, the ratios of the two Chern numbers c_1^2/c_2 of such surfaces can be any rational number between 1 and 2 (cf. [Z]).

In this note we shall give another partial answer to the conjecture:

Theorem. *Let (M, g) be a compact Kähler-Einstein surface with nonpositive bisectional curvature. Suppose $c_1 < 0$ and M is not quasi-ample. Then it is a quotient of the bidisc (hence, g is the canonical metric).*

2. Preliminaries

In this section we shall analyze the quasi ampleness condition. For our purpose we will only consider the dimension-two case. However, the higher-dimensional situations are similar.

We shall always assume that (M, g) is a general type Kähler surface with nonpositive bisectional curvature. Then $c_1(M) < 0$, as M cannot contain any rational curves.

Denote by $\pi: P = \mathbf{P}(T_M) \to M$ and L the dual of the tautological line bundle on P (such that $\pi_*(L) = \Omega_M$ is the cotangent bundle). Next let $\hat{g}$ be the hermitian metric on L induced by g, and write $Z_g = \{(x, [v]) \in P | \exists w \neq 0: R_{v\bar{v}w\bar{w}} = 0\}$ for the zero locus of the bisectional curvature of g. It is the subset where the nonnegative curvature form $c_1(L, \hat{g})$ fails to be positive definite. A surface $Y \subseteq P$ will be called *horizontal* if $\pi(Y) = M$.

Since $c_1(L, \hat{g})$ is always positive in the fiber direction of π, it follows that:

Lemma 1. *Let (M, g) be a general type Kähler surface with nonpositive bisectional curvature. If Y is a horizontal surface with $L^2 \cdot Y = 0$, then $Y \subset Z_g$.*

We shall also need the following lemma. Let C_1, C_2 be the two Chern forms under g, and let c_1, c_2 be the Chern classes. The nonpositivity of the bisectional curvature implies that $C_1^2 - C_2 \geq 0$ on M. Write $V = V_g = \{x \in M | C_1^2(x) - C_2(x) > 0\}$. V is not empty if and only if $c_1^2 > c_2$.

Lemma 2. *For $x \in V$ the set $\pi^{-1}(x) \cap Z_g$ consists of at most two points.*

Proof. Consider the Nakano tensor N defined by $N(\alpha \otimes \beta, \gamma \otimes \delta) = R_{\alpha\bar{\gamma}\beta\bar{\delta}}$ and extended linearly to $T_M \otimes T_M$. Since g is Kählerian, N is actually a hermitian bilinear form living in $S^2 T_M$. Since every element in $S^2 T_{x,M}$ is decomposable, the nonpositivity of bisectional curvature implies that $N \leq 0$. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq 0$ be the three eigenvalues of N. It is straightforward to show that $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = C_1^2 - C_2$. So, for $x \in V$, $\lambda_1 \leq \lambda_2 < 0$ at x. Hence, there can be at most one pair of directions at x which gives zero bisectional curvature. Q.E.D.

3. Proof of the Theorem

From now on we will assume that (M, g) is a compact Kähler-Einstein surface with $c_1 < 0$ and with nonpositive bisectional curvature. The two Chern numbers of M satisfy $3c_2 \geq c_1^2 \geq c_1 > 0$.

Lemma 3. *For the above M, $c_1^2 > c_2$.*

Proof. Assume that $c_1^2 = c_2$. Then since the Ricci curvature is everywhere negative, by Theorem A of [YZ], (M, g) is locally isometric to a hypersurface in $\mathbf{C}^3$. Since g is Einstein, the theorem of Smyth [S] implies that M is locally symmetric, which contradicts our assumption $c_1^2 = c_2$. Q.E.D.

Corollary. *If M is not quasi-ample, then for any $x \in V$ there exists a unique pair of directions $[\alpha]$, $[\beta]$ at x such that $R_{\alpha\bar{\alpha}\beta\bar{\beta}} = 0$. Moreover, α is perpendicular to β.*

Proof. The existence of such a pair is guaranteed by $\pi(Z_g) = M$ (Lemma 1), while the uniqueness comes from Lemma 2. Now suppose $R_{\alpha\bar{\alpha}\beta\bar{\beta}} = 0$. Let $[\alpha']$ and $[\beta']$ be the directions at x perpendicular to $[\alpha]$ and $[\beta]$, respectively. The Einstein condition implies that $R_{\alpha'\bar{\alpha}'\beta'\bar{\beta}'} = 0$. So for $x \in V$ these two pairs must coincide. Hence $\alpha \perp \beta$. Q.E.D.

Now we are ready to prove the theorem stated in §1.

Proof of the Theorem. Let Y be a horizontal surface with $L^2 \cdot Y = 0$. Then $Y \subseteq Z_g$. So by Lemma 2 the degree d of the restriction map $\pi|_Y \to M$ is one or two. First let us assume that $d = 1$.

Note that the metric g is analytic, so V is an open dense subset of M. Since $d = 1$, Y is a blowing up of M at finitely many points $E = \{p_1, \dots, p_r\}$. For any $x \in M \backslash E$ choose a holomorphic tangent frame (e_1, e_2) with $[e_1] \in Y$ in a neighborhood of x. Let (α, β) be a unitary frame near x with $[\alpha] = [e_1]$. Now $f = R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \equiv a$ is globally defined in $M \backslash E$, where a is the Ricci curvature. By the computation in [SY], $\Delta f = |R_{\alpha\bar{\beta}\alpha\bar{\beta}}|^2$ (f is $S_{1\bar{1}1\bar{1}}$ in [SY]; here we used the fact $[\alpha] = [e_1]$). So $R_{\alpha\bar{\beta}\alpha\bar{\beta}} = 0$. Then it follows that the connection and hence, the holonomy split (note that g is analytic). So (M, g) is reducible. Each de Rham factor of the universal covering space is again Einstein with negative Ricci curvature and hence, the Poincaré disc.

Next let us assume $d = 2$. Again let

$$E = \{x \in M | \pi^{-1}(x) \subseteq Y\}.$$

E is finite as Y is irreducible and horizontal. Since, for $x \in V$, $(\pi|_Y)^{-1}(x)$ consists of two perpendicular directions and V is dense in M, it follows that $\pi|Y$ cannot have any branch locus over $M \backslash E$. Hence it gives a 2-sheets unbranched covering over $M \backslash E$ and, hence, an unbranched double cover $M' \to M$. Now by the branched covering trick, in the pullback of the fiber bundle $\pi: P \to M$ by $M' \to M$, the inverse image of Y consists of two irreducible components, each with degree one over M', so we are in the first case (or it gives a holomorphic splitting of $T_{M'}$ in $M' \backslash E'$; therefore, $T_{M'}$ is not stable and $M' = D \times D/\Gamma$). Q.E.D.

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