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APPROXIMATION OF NORMAL ELEMENTS IN THE MULTIPLIER ALGEBRA OF AN AF C^*-ALGEBRA

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ABSTRACT. It is shown that there is a simple separable AF algebra A such that $M(\mathscr{K} \otimes A)$ does not have weak (FN) and such that the generalized Berg-Weyl-von Neumann Theorem does not hold for $\mathscr{K} \otimes A$.

Consider two properties enjoyed by every selfadjoint element h of $L(H) = M(\mathscr{K})$.

(1) h can be approximated in norm by selfadjoint elements with finite spectrum.

(2) There is a sequence of mutually orthogonal compact projections (e_n) and a bounded sequence of real numbers (λ_n) such that $h - \sum_{n=1}^{\infty} \lambda_n e_n \in \mathscr{K}$ (the Weyl-von Neumann Theorem).

A C^*-algebra is said to be of *real rank zero* if (1) holds for every selfadjoint element h. The generalized Weyl-von Neumann Theorem for a C^*-algebra A states that (2) holds for every selfadjoint element in the multiplier algebra $M(A)$, where $\mathscr{K}$ is replaced by A (and the projections (e_n) belong to A and sum to the identity). These generalizations of (1) and (2) have attracted much attention in recent years. In particular, when A is σ-unital and has real rank zero, it has been shown that $M(A)$ has real rank zero if and only if the generalized Weyl-von Neumann Theorem holds for A [Lin1, Zha]. Moreover, Lin has shown that if A is a σ-unital AF algebra, then $M(A)$ has real rank zero [Lin2]. As AF algebras are in many ways the simplest generalization of the algebra $\mathscr{K}$, this result is very encouraging.

Consider now analogues of properties (1) and (2) for normal operators. A C^*-algebra is said to have property (FN) if every normal element can be approximated in norm by normal elements having finite spectrum. By the spectral theorem, $L(H)$ has (FN). In more general multiplier algebras this definition is too strong. In a C^*-algebra whose K_1 group is nontrivial, an element x might fail to be approximated by elements with finite spectrum due to index obstructions corresponding to holes in the spectrum of x. Lin has defined *weak* (FN) to take such obstructions into account [Lin3, Definition 4.2]. Without recalling

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the definition, we remark that if an algebra has weak (FN) then a normal element with spectrum equal to a disk can be approximated by normal elements with finite spectrum.

Berg's generalization of the Weyl-von Neumann Theorem states that (2) holds for every normal operator h in $L(H)$, where the (λ_n) are allowed to be complex [Ber]. There are several generalizations of this to multiplier algebras. We will use the following

Definition. Let A be a C^*-algebra. We say that the generalized Berg-Weyl-von Neumann Theorem holds for A if the normal elements of $M(A)$ are *quasi-diagonal* (cf. [Zha, 1.3]); i.e., given any normal element h in $M(A)$ there is a sequence of mutually orthogonal projections (e_n) in A and a bounded sequence (a_n) in A such that

(i) $\sum e_n = 1$,
(ii) $a_n = e_n a_n e_n$ for all n,
(iii) $h - \sum a_n \in A$,

where the sums in (i) and (iii) are taken in the strict topology.

The result of this note is to point out the following

Theorem 1. *There is a simple separable AF algebra A such that $M(\mathscr{K} \otimes A)$ does not have weak (FN) and such that the generalized Berg-Weyl-von Neumann Theorem does not hold for $\mathscr{K} \otimes A$.*

We first prove the more general

Theorem 2. *Let A be a C^*-algebra admitting a *-homomorphism $\varphi\colon C_0(\mathbf{R}^2) \to A$ such that $\varphi_*\colon K_0(C_0(\mathbf{R}^2)) \to K_0(A)$ is nonzero. Then $M(\mathscr{K} \otimes A)$ does not have weak (FN) and the generalized Berg-Weyl-von Neumann Theorem does not hold for $\mathscr{K} \otimes A$.*

Proof. By Theorems 1 and 2 of [MS] there is a normal element $h \in M(\mathscr{K} \otimes A)$ such that the spectrum of h is the closed unit disk, $\pi(h)$ is unitary in $M(\mathscr{K} \otimes A)/\mathscr{K} \otimes A$, and $\partial[\pi(h)] = \varphi_*(b)$, where π is the quotient map, ∂ is the connecting map in K-theory, and b is the generator of $K_0(C_0(\mathbf{R}^2))$. It follows that $\pi(h)$ cannot be norm-approximated by invertibles with finite spectrum and, hence, that h cannot be norm-approximated by elements with finite spectrum. As remarked earlier, this implies that $M(\mathscr{K} \otimes A)$ does not have weak (FN).

Now suppose that the generalized Berg-Weyl-von Neumann Theorem holds for $\mathscr{K} \otimes A$. Let (e_n) and (a_n) be as in the above definition for the element h. Let $x = h - \sum a_n$. Since $\pi(h)$ is unitary, we have

$$\sum (e_n - a_n^* a_n) = 1 - (h-x)^*(h-x) \in \mathscr{K} \otimes A$$

and, similarly, $\sum (e_n - a_n a_n^*) \in \mathscr{K} \otimes A$. Therefore, $\|e_n - a_n^* a_n\| \to 0$ and $\|e_n - a_n a_n^*\| \to 0$. It follows that, for large enough n, $y_n = \sum_{k=1}^n e_k + \sum_{k=n+1}^\infty a_k$ is invertible. Since $h - y_n \in \mathscr{K} \otimes A$, we obtain the contradiction

$$\partial[\pi(h)] = \partial[\pi(y_n)] = \partial \circ \pi_*[y_n] = 0. \quad \square$$

Proof of Theorem 1. Elliott and Loring have shown that a simple unital AF algebra admits a *-homomorphism φ as in the statement of Theorem 2 if and only if its dimension group contains nonzero elements in the intersection of the kernels of all the finite traces [EL]. Simple dimension groups containing such elements abound and provide examples verifying Theorem 1.

Specifically, they have a quite simple explicit example, which appears in §6 of [Lor1]. The stationary inductive system given by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ defines a simple AF algebra whose dimension group is $\{(m/3^n, k) | m \equiv k \pmod 2\} \subseteq \mathbf{Z}[\frac{1}{3}] \oplus \mathbf{Z}$, with strict order from the first coordinate. □

Remark. For an example of a separable C^*-algebra having real rank zero but not having weak (FN), see [Lor2].

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