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ON A COMPLEMENTARITY PROBLEM IN BANACH SPACE

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ABSTRACT. The paper gives a result on the existence and uniqueness of solutions for the complementarity problems associated with hemicontinuous monotone mappings of convex cones. We correct the assumptions and proof given in an earlier paper of A. T. Dash and S. Nanda [*A complementarity problem in mathematical programming in Banach space*, J. Math. Anal. Appl. **98** (1984), 328–331].

1. INTRODUCTION AND STATEMENT OF THE THEOREM

Let B be reflexive real Banach space, and let B^* be its dual. Let the value of $u \in B^*$ at $x \in B$ be denoted by (u, x). Let C be a closed convex cone in B with the vertex at 0. The polar of C is the cone C^* defined by

$$C^* = \{u \in B^* \colon (u, x) \geq 0 \text{ for each } x \in C\}.$$

For any $e \in C^*$ and $r > 0$ we write

$$\begin{aligned} D_r(e) &= \{x \in C \colon 0 \leq (e, x) \leq r\}, \\ D_r^0(e) &= \{x \in C \colon 0 < (e, x) < r\}, \\ S_r(e) &= \{x \in C \colon (e, x) = r\}. \end{aligned}$$

A mapping $T \colon C \to B^*$ is said to be monotone if

$$(Tx - Ty, \ x - y) \geq 0$$

for all $x, y \in C$ and strictly monotone if strict inequality holds whenever $x \neq y$. The mapping T is said to be hemicontinuous on C if for all $x, y \in C$ the map $t \mapsto T(ty+(1-t)x)$ of $[0, 1]$ to B^* is continuous when B is endowed with the weak* topology. The mapping T is said to be bounded if T maps bounded subsets of C into bounded subsets of B^*.

The purpose of this note is to prove the following existence and uniqueness theorem for the nonlinear complementarity problem.

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Theorem. *Let $T: C \to B^*$ be hemicontinuous and monotone such that there is an $x \in C$ with $Tx \in \operatorname{int} C^*$. Then there is an x_0 such that*

$$x_0 \in C, \qquad Tx_0 \in C^*, \qquad (Tx_0, x_0) = 0. \tag{1.1}$$

Further, if T is strictly monotone, then there is unique x_0 satisfying (1.1).

This work has been motivated by the work of Dash and Nanda [2], who have incorrectly proved the same result under an additional hypothesis of boundedness of the operator T but under the weaker assumption that there exists an $x \in C$ with $Tx \in C^*$. We shall show that the result of Dash and Nanda [2] is not true (even with bounded T) and that the stronger assumption $Tx \in \operatorname{int} C^*$ cannot be omitted.

To prove the theorem we need the following result of Browder (see Browder [1] and Mosco [3]).

Proposition. *Let T be a monotone hemicontinuous map of a closed, convex, bounded subset K of B, with $0 \in K$, into B^*. Then there is an $x_0 \in K$ such that $(Tx_0, y - x_0) \geq 0$ for all $y \in K$.*

2. Proof of the Theorem

For any $e \in \operatorname{int} C^*$ and each $r > 0$, $D_r(e)$ is clearly convex. The function $f: C \to R$ defined by $f(z) = (e, z)$ is obviously continuous. Since $D_r(e) = f^{-1}[0, r]$, the set $D_r(e)$ is closed. We now need to show that $D_r(e)$ is bounded. Suppose to the contrary that it is not. Then we can choose a sequence $\{z_n\}$ of isolated points in $D_r(e)$ satisfying

$$\|z_n\| \to \infty \quad \text{as } n \to \infty.$$

Let $y_n = z_n/\|z_n\|$. Then $\|y_n\| = 1$ and $y_n \in D_r(e)$ for all n. We therefore have a weakly convergent subsequence $y_{n_k} \rightharpoonup y$. Since $D_r(e)$ is closed, $y \in D_r(e)$. Moreover,

$$(e, y) = \lim_{k \to \infty} (e, y_{n_k}) = \lim_{k \to \infty} (e, z_{n_k}/\|z_{n_k}\|).$$

But

$$(e, z_{n_k}/\|z_{n_k}\|) \leq r/\|z_{n_k}\| \to 0 \quad \text{as } k \to \infty.$$

Thus $(e, y) = 0$. Since $e \in \operatorname{int} C^*$, we conclude that $y = 0$. This is a contradiction in view of the fact that $\|y_{n_k}\| = 1$ for every k.

In summary, $D_r(e)$ is closed, convex, and bounded for each $r > 0$. Therefore, it follows from the above proposition that for each $r > 0$ there is an $x_r \in D_r(e)$ such that

$$(Tx_r, y - x_r) \geq 0 \quad \text{for all } y \in D_r(e). \tag{2.1}$$

Since $0 \in D_r(e)$, it follows that $(Tx_r, x_r) \leq 0$. If there exist $e \in \operatorname{int} C^*$ and $r > 0$ such that $x_r \in D_r^0(e)$, then there is some $\lambda > 1$ such that $\lambda x_r \in S_r(e) \subset D_r(e)$. Then from (2.1) we have that $(Tx_r, x_r) \leq (Tx_r, \lambda x_r) = \lambda(Tx_r, x_r)$. Since $(Tx_r, x_r) \leq 0$, it is impossible unless $(Tx_r, x_r) = 0$; thus x_r satisfies (1.1). Now assume that $x_r \in S_r(e)$ for all $e \in \operatorname{int} C^*$ and all $r > 0$. By the hypothesis there is an $x \in C$ with $Tx \in \operatorname{int} C^*$. Set $e = Tx$. Choose $r > (Tx, x) \geq 0$. Now $x \in D_r^0(Tx)$, and since T is monotone, we have

$$\begin{aligned} &(Tz, z - x) \geq (Tx, z - x) > 0 \quad \text{for all } z \in S_r(Tx), \\ &\text{but } x_r \in S_r(Tx);\ \text{hence } (Tx_r, x_r - x) > 0. \end{aligned} \tag{2.2}$$

Since $x \in D_r^0(Tx) \subset D_r(Tx)$, it follows from (2.1) that $(Tx_r, x - x_r) \geq 0$. Since this contradicts (2.2), the assumption that $x_r \in S_r(e)$ for all r has thus been shown not to hold when $e = Tx$. Thus the proof of the theorem is reduced to the previous case, the case where there exists $e \in \operatorname{int} C^*$ and $r > 0$ such that $x \in D_r^0(e)$. If T is strictly monotone, it is easy to see that the solution is unique, and this completes the proof of the theorem.

Now observe that if $e \in C^*$ but $e \notin \operatorname{int} C^*$, the sets $D_r(e)$ need not be bounded. In this case we cannot conclude $y = 0$ from the fact that $(e, y) = 0$. Consider the case when $B = R^2$, $C = R_+^2$, and $e = (1, 0)$. Then, for each $r > 0$, $D_r(e)$ contains the positive y-axis and hence is unbounded. Therefore, Proposition (Browder) cannot be applied in this case. Also we cannot apply Lemma (Mosco) (see Mosco [4] as it was done in [2], because Lemma holds for nonempty closed convex and bounded sets in C.

In fact, the following example shows that the theorem of [2] is false. The counterexample was suggested to the author by one of the referees of *Zeitschrift Mathematische Operations-forschung und Statistik-Series Optimization* in connection with some other paper and was communicated to the author by Professor Dr. K.-H. Elster, the editor.

Take $B = R^3$ and $C = \{(x, y, z) \in R^3 : x, z \geq 0, 2xz \geq y^2\}$. Define T by $T(x, y, z) = (x+1, y+1, 0)$. Then T is monotone and hemicontinuous (even bounded). The point $(1, -1, 1) \in C$ and $T(1, -1, 1) = (2, 0, 0) \in C^*$. If $u = (x, y, z) \in C$ with $Tu \in C^*$, then $y = -1$, and hence $x > 0$. Hence, for any such u, $(Tu, u) = x(x+1) > 0$.

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