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COMMON SUBSPACES OF L_p-SPACES

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ABSTRACT. For $n \geq 2$, $p < 2$, and $q > 2$ does there exist an n-dimensional Banach space different from Hilbert spaces which is isometric to subspaces of both L_p and L_q ? Generalizing the construction from the paper *Zonoids whose polars are zonoids* by R. Schneider (Proc. Amer. Math. Soc. **50** (1975), 365–368) we give examples of such spaces. Moreover, for any compact subset Q of $(0, \infty)\backslash\{2k, k \in N\}$ we can construct a space isometric to subspaces of L_q for all $q \in Q$ simultaneously.

1. INTRODUCTION

This work started with the following question: for given $n \geq 2$, $p \in (0, 2)$, and $q > 2$ does there exist an n-dimensional Banach space which is different from Hilbert spaces and isometric to subspaces of both L_p and L_q ?

It is a well-known fact first noticed by P. Levy that Hilbert spaces are isometric to subspaces of L_q for all $q > 0$. On the other hand, it was proved in [4] that for $n \geq 3$, $q > 2$, and $p > 0$ the function $\exp(-\|x\|_q^p)$ is not positive definite where $\|x\|_q = (|x_1|^q + \cdots + |x_n|^q)^{1/q}$. (This result gave an answer to a question posed by Schoenberg [12] in 1938.) In 1966 Bretagnolle, Dacunha-Castelle, and Krivine [1] proved that for $0 < p < 2$ a space $(E, \|\cdot\|)$ is isometric to a subspace of L_p if and only if the function $\exp(-\|x\|^p)$ is positive definite. Thus, in the language of isometries, the above-mentioned result from [4] means that for every $n \geq 3$, $q > 2$, and $p \in (0, 2)$ the space l_n^q is not isometric to a subspace of L_p. (For $p \geq 1$ this fact was first proved in [2].) The initial purpose of this work was to find a non-Hilbertian subspace $(E, \|\cdot\|)$ of L_q with $q > 2$ of dimension at least 3 such that the function $\exp(-\|x\|^p)$ is positive definite. The latter problem is equivalent to that at the beginning of the paper.

We prove, however, a more general fact: for every $n \geq 2$ and every compact subset Q of $(0, \infty)\backslash\{2k, k \in N\}$ there exists an n-dimensional Banach space different from Hilbert spaces which is isometric to subspaces of L_q for all $q \in Q$ simultaneously.

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In 1975 Schneider [11] proved that there exist nontrivial zonoids whose polars are zonoids or, in other words, there exist non-Hilbertian Banach spaces X such that X and X^* are isometric to subspaces of L_1. It turns out that Schneider's construction of special subspaces of L_1 can be extended to all numbers $q > 0$ which are not even integers, and in this way we obtain our main result.

2. Some properties of spherical harmonics

We start with some properties of spherical harmonics (see [6] for details).

Let P_m denote the space of spherical harmonics of degree m on the unit sphere Ω_n in R^n. Recall that spherical harmonics of degree m are restrictions to the sphere of harmonic homogeneous polynomials of degree m. We consider spherical harmonics as functions from the space $L_2(\Omega_n)$. Any two spherical harmonics of different degrees are orthogonal in $L_2(\Omega_n)$ [6, p. 2].

The dimension $N(n, m)$ of the space P_m can easily be calculated [6, p. 4]:

$$N(n, m) = \frac{(2m+n-2)\Gamma(n+m-2)}{\Gamma(m+1)\Gamma(n-1)}. \tag{1}$$

Let $\{Y_{mj}\colon j = 1, \dots, N(n, m)\}$ be an orthonormal basis of the space P_m. By the Addition Theorem [6, p. 9], for every $x \in \Omega_n$,

$$\sum_{j=1}^{N(n,m)} Y_{mj}^2(x) = \frac{N(n, m)}{\omega_n}, \tag{2}$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the sphere Ω_n.

Linear combinations of functions Y_{mj} are dense in the space $L_2(\Omega_n)$ [6, p. 43]. Therefore, if F is a continuous function on Ω_n and $(F, Y_{mj}) = 0$ for every $m = 0, 1, 2, \dots$ and every $j = 1, \dots, N(n, m)$, then $F \equiv 0$ on Ω_n. Here (F, Y) stands for the scalar product in $L_2(\Omega_n)$.

Let Δ be the Laplace-Beltrami operator on the sphere Ω_n. Then for every $Y_m \in P_m$ we have [6, p. 39]

$$\Delta Y_m + m(m+n-2)Y_m \equiv 0. \tag{3}$$

An immediate consequence of (3) (and a well-known fact) is that Δ is a symmetric operator, and we can apply Green's formula: for every function H from the class C^{2r}, $r \in N$, of functions on Ω_n having continuous partial derivatives of order $2r$ and for every $Y_m \in P_m$, $m \geq 1$,

$$(-m(m+n-2))^r(H, Y_m) = (H, \Delta^r Y_m) = (\Delta^r H, Y_m). \tag{4}$$

We also need the Funk-Henke formula [6, p. 20]: for every $Y_m \in P_m$, every continuous function f on $[-1, 1]$, and every $x \in \Omega_n$

$$\int_{\Omega_n} f(\langle x, \xi\rangle)Y_m(\xi)\,d\xi = \lambda_m Y_m(x), \tag{5}$$

where $\langle\cdot, \cdot\rangle$ stands for the scalar product in R^n and

$$\lambda_m = \frac{(-1)^m\pi^{(n-1)/2}}{2^{m-1}\Gamma(m+(n-1)/2)} \int_{-1}^{1} f(t)\frac{d^m}{dt^m}(1-t^2)^{m+(n-3)/2}\,dt. \tag{6}$$

Let us calculate λ_m in the case where $f(t) = |t|^q$, $q > 0$. The numbers λ_m have already been calculated by Richards [8]. However, we present a simple proof to make this paper complete.

Lemma 1. *If m is an even integer, $q > 0$, $q \neq 2k$, $k \in N$, and $f(t) = |t|^q$, then*

$$\lambda_m = \frac{\pi^{n/2-1}\Gamma(q+1)\sin(\pi(m-q)/2)\Gamma((m-q)/2)}{2^{q-1}\Gamma((m+n+q)/2)}. \tag{7}$$

Note that $\lambda_m = 0$ for odd m.

Proof. First assume that $q > m$, and calculate the integral from (6) by parts m times. Then use the formula $\int_{-1}^{1} |t|^{2\alpha-1}(1-t^2)^{\beta-1}\,dt = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ and formulas for the Γ-function: $\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})/\pi^{1/2}$ and $\Gamma(1-x)\Gamma(x) = \pi/\sin(\pi x)$. We get (7) for $q > m$. Note that both sides of (7) are analytic functions of q in the domain $\Re q > 0$, $q \neq 2k$, $k \in N$. Because of the uniqueness of analytic extension, (7) holds for every q from this domain. We are done. □

3. Main result

Let X be an n-dimensional subspace of $L_q = L_q([0,1])$ with $q > 0$. Let $f_1, \dots, f_n$ be a basis in X and μ the joint distribution of the functions $f_1, \dots, f_n$ with respect to Lebesgue measure (μ is a finite measure on R^n). Then for every $x \in R^n$

$$\begin{aligned} \|x\|^q &= \left\|\sum_{k=1}^{n} x_k f_k\right\|^q = \int_0^1 \left|\sum_{k=1}^{n} x_k f_k(t)\right|^q dt \\ &= \int_{R^n} |\langle x, \xi\rangle|^q \, d\mu(\xi) = \int_{\Omega_n} |\langle x, \xi\rangle|^q \, d\nu(\xi), \end{aligned} \tag{8}$$

where ν is the projection of μ to the sphere. (For every Borel subset A of Ω_n, $\nu(A) = \int_{\{tA,\, t\in R\}} \|x\|_2^q \, d\mu(x)$.) Representation (8) of the norm is usually called the Levy representation. It is clear now that a norm in an n-dimensional Banach space admits the Levy representation with a probability measure on the sphere if and only if this space is isometric to a subspace of L_q. (Given the Levy representation we can choose functions $f_1, \dots, f_n$ on $[0,1]$ with the joint distribution ν and define an isometry by $x \to \sum_{k=1}^{n} x_k f_k$, $x \in R^n$.)

If we replace the measure ν by an arbitrary continuous (not necessarily nonnegative) function on the sphere, then a representation similar to the Levy representation is possible for a large class of Banach spaces (see [5] for the Levy representation with distributions instead of measures; such a representation is possible for any Banach space and any q which is not an even integer). This is an idea going back to Blaschke that any smooth enough function on the sphere can be represented in the form (8) with a continuous function instead of a measure on the sphere. However, Blaschke and then Schneider [10] restricted themselves to the case $q = 1$ which is particularly important in the theory of convex bodies. The following theorem is an extension of Schneider's results from [10, p. 77] and [11, p. 367] to all positive numbers q which are not even integers. Note that for $q \in (0, 2)$ representation (9) was obtained and applied to determining spectral measures of stable laws by Richards in [9].

Theorem 1. *Let $q > 0$, $q \neq 2k$, $k \in N$, and let H be an even function of the class C^{2r} on Ω_n, where $r \in N$ and $2r > n + q$. Then there exists a continuous* *function* b_H *on the sphere* Ω_n *such that for every* $x \in \Omega_n$

$$H(x) = \int_{\Omega_n} |\langle x, \xi\rangle|^q b_H(\xi)\, d\xi. \tag{9}$$

Besides that, there exist constants $K(q)$ *and* $L(q)$ *depending on* n *and* q *only such that for every* $x \in \Omega_n$

$$|b_H(x)| \le K(q)\|H\|_{L_2(\Omega_n)} + L(q)\|\Delta^r H\|_{L_2(\Omega_n)}. \tag{10}$$

Proof. Define a function b_H on Ω_n by

$$b_H(x) = \sum_{m=0;\,2|m}^{\infty} \lambda_m^{-1} \sum_{j=1}^{N(n,m)} (H, Y_{mj})Y_{mj}(x). \tag{11}$$

Let us prove that the series in the right-hand side of (11) converges uniformly on Ω_n. By the Cauchy-Schwartz inequality, (2), and the fact that Y_{mj} form an orthonormal basis in P_m, we get

$$\begin{aligned}\left|\sum_{j=1}^{N(n,m)} (\Delta^r H, Y_{mj})Y_{mj}(x)\right| &\le \left(\sum_{j=1}^{N(n,m)} (\Delta^r H, Y_{mj})^2\right)^{1/2} \left(\sum_{j=1}^{N(n,m)} Y_{mj}^2(x)\right)^{1/2} \\ &\le \|\Delta^r H\|_{L_2(\Omega_n)} \left(\frac{N(n,m)}{\omega_n}\right)^{1/2}.\end{aligned}$$

It follows from (4) and the latter inequality that

$$\begin{aligned}|b_H(x)| &\le |\lambda_0^{-1}(H, Y_0)Y_0(x)| \\ &\quad + \sum_{m=2;\,2|m}^{\infty} \lambda_m^{-1} \left(\frac{-1}{m(m+n-2)}\right)^r \left|\sum_{j=1}^{N(n,m)} (\Delta^r H, Y_{mj})Y_{mj}(x)\right| \\ &\le |\lambda_0|^{-1}\omega_n^{-1/2}\|H\|_{L_2(\Omega_n)} + \sum_{m=2;\,2|m}^{\infty} \lambda_m^{-1} m^{-2r} \left(\frac{N(n,m)}{\omega_n}\right)^{1/2} \|\Delta^r H\|_{L_2(\Omega_n)}.\end{aligned}$$

Let us show that the series $\sum_{m=2;\,2|m}^{\infty} \lambda_m^{-1} m^{-2r} (N(n,m)/\omega_n)^{1/2}$ converges. In fact, it follows from (1) that $N(n,m) = O(m^{n-2})$, and it is an easy consequence of (7) and the Stirling formula that $\lambda_m^{-1} = O(m^{(n+2q)/2})$. Since $2r > n+q = (n+2q)/2 + (n-2)/2 + 1$, we get $\lambda_m^{-1} m^{-2r}(N(n,m)/\omega_n)^{1/2} = o(m^{-1-\varepsilon})$ for some $\varepsilon > 0$, and the series is convergent. We denote the sum of this series by $L(q)$ and put $K(q) = |\lambda_0|^{-1}\omega_n^{-1/2}$, so we get (10).

We have proved that the series in (11) converges uniformly and defines a continuous function on Ω_n. It follows from (5) and the fact that all functions Y_{mj} are orthogonal that $(H, Y_{mj}) = (\int_{\Omega_n} |\langle x, \xi\rangle|^q b_H(\xi)\, d\xi,\ Y_{mj}(x))$ for every $m = 0, 1, 2, \dots$ and $j = 1, \dots, N(n,m)$. Hence, the function b_H satisfies (9). □

Let X be an n-dimensional Banach space, and let $q > 0$, $q \ne 2k$, $k \in N$. Let $c(q) = \Gamma((n+q)/2)/(2\Gamma((q+1)/2)\pi^{(n-1)/2})$ be a constant such that $1 = c(q)\int_{\Omega_n} |\langle x, \xi\rangle|^q\, d\xi$ for every $x \in \Omega_n$. (The latter integral does not depend on the choice of $x \in \Omega_n$; it means that the norm of the space l_2^n admits the Levy representation with the uniform measure on the sphere and the space l_2^n is isometric to a subspace of L_q for every q.)

Denote by $H(x)$, $x \in \Omega_n$, the restriction of the function $\|x\|^q$ to the sphere Ω_n. Assume that the function H belongs to the class C^{2r} on Ω_n, where $2r > n+q$, $r \in N$. Let b_H be the function corresponding to H by Theorem 1.

Lemma 2. *If the number* $K(q)\|H-1\|_{L_2(\Omega_n)} + L(q)\|\Delta^r H\|_{L_2(\Omega_n)}$ *is less than* $c(q)$, *then the space* X *is isometric to a subspace of* L_q.

Proof. By (9) and definition of the number $c(q)$,

$$H(x) - 1 = \int_{\Omega_n} |\langle x, \xi\rangle|^q (b_H(\xi) - c(q))\, d\xi$$

for every $x \in \Omega_n$. By (9), $|b_H(x) - c(q)| < c(q)$ for every $x \in \Omega_n$. It means that the function b_H is positive on the sphere. Equality (9) implies that the space X admits the Levy representation with a nonnegative measure, and, by the reasoning at the beginning of §3, X is isometric to a subspace of L_q. □

Now we are able to prove the main result of this paper. Let us only note that for every function f of the class C^2 on the sphere Ω_n and for a small enough number λ the function $N(x) = 1 + \lambda f(x)$, $x \in \Omega_n$, is the restriction to the sphere of some norm in R^n. This is an easy consequence of the following one-dimensional fact: If $a, b \in R$, g is a convex function on $[a, b]$ with $g'' > \delta > 0$ on $[a, b]$ for some δ and $h \in C^2[a, b]$, then functions $g + \lambda h$ have positive second derivatives on $[a, b]$ for sufficiently small λ's and, hence, are convex on $[a, b]$.

Theorem 2. *Let* Q *be a compact subset of* $(0, \infty)\backslash\{2k, k \in N\}$. *Then there exists a Banach space different from Hilbert spaces which is isometric to a subspace of* L_q *for every* $q \in Q$.

Proof. Let f be any infinitely differentiable function on Ω_n, and fix a number $r \in N$ so that $2r > n+q$ for every $q \in Q$. Choose a sufficiently small number λ such that the function $N(x) = 1 + \lambda f(x)$, $x \in \Omega_n$, is the restriction to the sphere of some norm in R^n (see the remark before Theorem 2) and such that for every $q \in Q$ the function $H(x) = (N(x))^q$ satisfies the condition of Lemma 2. The possibility of such a choice of λ follows from the facts that $K(q)$, $L(q)$, and $c(q)$ are continuous functions of q on the set Q and that $\|H-1\|_{L_2(\Omega_n)}$ and $\|\Delta^{2r} H\|_{L_2(\Omega_n)}$ tend to zero uniformly with respect to $q \in Q$ as λ tends to zero. Now we can apply Lemma 2 to complete the proof. □

Finally, let us consider the case where q is an even integer. It is easy to see that for any fixed number $2k$, $k \in N$, $k > 1$, we can make the space X constructed in Theorem 2 isometric to a subspace of L_{2k}. In fact, let $N(x) = (1+\lambda(x_1^{2k}+\cdots+x_n^{2k}))^{1/2k}$. For sufficiently small numbers λ, N is the restriction to the sphere of some norm in R^n and the corresponding space X is isometric to a subspace of L_q for every $q \in Q$. On the other hand, X is isometric to a subspace of L_{2k} because the norm admits the Levy representation with a measure on the sphere

$$1 + \lambda(x_1^{2k} + \cdots + x_n^{2k}) = \int_{\Omega_n} |\langle x, \xi\rangle|^{2k} (c(2k)\, d\xi + \lambda\, d\delta_1(\xi) + \cdots + \lambda\, d\delta_n(\xi)),$$

where δ_i is a unit mass at the point $\xi \in R_n$ with $\xi_i = 1$, $\xi_j = 0$, $j \neq i$.

Let us show that one cannot make the space X isometric to subspaces of L_{2p} and L_{2q} if $p, q \in N$ and do not have common factors. In fact, if $(X, \|\cdot\|)$ is such a space, then for every $x \in R_n$

$$\|x\|^{4pq} = \left(\int_{\Omega_n} |\langle x, \xi\rangle|^{2p}\, d\mu(\xi)\right)^{2q} = \left(\int_{\Omega_n} |\langle x, \xi\rangle|^{2q}\, d\nu(\xi)\right)^{2p}$$

for some measures μ, ν on Ω_n. The functions in the latter equality are polynomials, and since the polynomial ring has the unique factorization property, we conclude that $\|x\|^2$ is a homogeneous polynomial of the second order and X is a Hilbert space.

The situation is not clear if p and q have common factors. One can find some interesting results on Banach spaces with polynomial norms and on the structure of subspaces of L_{2k}, $k \in N$ [7].

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