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A LARGE Π_2^1 SET, ABSOLUTE FOR SET FORCINGS

SY D. FRIEDMAN

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ABSTRACT. We show how to obtain, by class-forcing over L, a set of reals X which is large in $L(X)$ and has a Π_2^1 definition valid in all set-generic extensions of $L(X)$. As a consequence we show that it is consistent for the Perfect Set Property to hold for Σ_2^1 sets yet fail for some Π_2^1 set. Also it is consistent for the perfect set property to hold for Σ_2^1 sets and for there to be a long Π_2^1 well-ordering. These applications (necessarily) assume the consistency of an inaccessible cardinal.

The purpose of this note is to prove the following.

Theorem. *Let κ be an L-cardinal, definable in L. Then there is a set of reals X, class-generic over L, such that*

(a) *$L(X) \models \mathrm{Card} = \mathrm{Card}^L$ and X has cardinality κ.*

(b) *Some fixed Π_2^1 formula defines X in all set-generic extensions of $L(X)$.*

By Lévy-Shoenfield Absoluteness, any Π_2^1 formula defining X in $L(X)$ defines a superset of X in each extension of $L(X)$. The point of (b) is that this superset is just X in set-generic extensions of $L(X)$. If $O^\#$ exists then X as in the Theorem actually exists in V, though of course it will be only countable there.

The basic idea of the proof comes from David [2]. In his paper a real R class-generic over L is produced so that $\{R\}$ is Π_2^1, uniformly for set-generic extensions of $L(R)$. The added technique here is to use "diagonal supports" to take a large product of David-style forcings.

The following corollaries are further applications of the Theorem and its proof.

Corollary 1. *Assume consistency of an inaccessible cardinal. Then it is consistent for the Perfect Set Property to hold for Σ_2^1 sets yet fail for some Π_2^1 set.*

Proof. Use the Theorem to obtain a Π_2^1 set X which has cardinality κ in $L(X)$, $\kappa =$ least L-inaccessible, and which has a Π_2^1-definition uniform for set-generic extensions. Then gently collapse κ to ω_1 and add ω_2 Cohen reals. In this extension, $\omega_1 > \omega_1^{L(R)}$ for each real R and X is a Π_2^1 set of cardinality $\omega_1 < \omega_2 = 2^{\aleph_0}$. □

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Corollary 2. *Assume consistency of an inaccessible. Then it is consistent that the Perfect Set Property holds for Σ^1_2 sets and there is a Π^1_2 well-ordering of some set of reals of length $\aleph_{1000}$.*

The latter answers a question of Harrington [4].

THE PROOF

We modify the construction of David [2] to suit our purposes. First we describe the α^+-Souslin tree T_α in L, where α is a successor L-cardinal: T_α has a unique node on level 0 and exactly two immediate successors on level $\beta+1$ to each node on level β, for $\beta < \alpha^+$. If $\beta < \alpha^+$ is a limit of cofinality $< \alpha$ then level β assigns a top to each branch through the tree below level β. Now suppose $\beta < \alpha^+$ has cofinality α. Let $\mathscr{P}$ be the forcing consisting of pairs (γ, f) where $\gamma < \beta$ and f is a function from γ into the nodes at levels $< \beta$, with extension defined by $(\gamma', f') \le (\gamma, f)$ iff $\gamma' \ge \gamma$, $f'(\delta)$ tree-extends $f(\delta)$ for each $\delta < \gamma$. Choose G to be $\mathscr{P}$-generic over L_{β^*} where $\beta^* =$ largest p.r. closed $\beta^* \ge \beta$ such that $\beta^* = \beta$ or $L_{\beta^*} \vDash \text{card}(\beta) > \alpha$. Then the nodes on level β are obtained by putting tops on the branches defined by $\{f(\delta) | (\gamma, f) \in G \text{ some } \gamma\}$ for $\delta < \beta$. This completes the definition of the α^+-Souslin tree T_α.

Now fix an L-definable cardinal κ and also fix an L-definable 1-1 function $F : \kappa \times \omega \times \text{ORD} \to$ Successor L-cardinals greater than κ. The forcing $\mathscr{P}(\gamma, n)$, $\gamma < \kappa$ and $n < \omega$, is designed to produce a real $R(\gamma, n)$ coding branches through T_α whenever α is of the form $F(\gamma, n, \delta)$ for some δ. This forcing is obtained by modifying the Jensen coding of the empty class (see Beller, Jensen, and Welch [1]) as follows: In defining the strings $s : [\alpha, |s|] \to 2$ in S_α, require that $\text{Even}(s)$ code a branch through T_α if $\alpha \in \text{Card}(\gamma, n) = \{F(\gamma, n, \delta) | \delta \in \text{ORD}\}$. Also use David's trick to create a Π^1_2 condition implying that branches through the appropriate trees are coded: for any α, for s to belong to S_α require that for $\xi \le |s|$ and $\eta > \xi$, if $L_\eta(s \restriction \xi) \vDash \xi = \alpha^+ + ZF^- + \text{Card} = \text{Card}^L$ then $L_\eta(s \restriction \xi) \vDash$ for some $\gamma^* < \kappa^*$, $\text{Even}(s \restriction \xi)$ codes a branch through $T^*_{\alpha^*}$ whenever $\alpha^* \in \text{Card}^*(\gamma^*, n)$, where κ^*, $T^*_{\alpha^*}$, $\text{Card}^*(\gamma^*, n)$ are defined in L_η as were κ, T_α, $\text{Card}(\gamma, n)$ in L. The $\le \alpha$-distributivity of $\mathscr{P}(\gamma, n)_\alpha$ (= the forcing at and above α) is established in David [2], with one added observation: if $\alpha' \in \text{Card}(\gamma, n)$ then we have to be sure that $\text{Even}(p_{\alpha'})$ codes a branch through $T_{\alpha'}$, where p arises as the greatest lower bound to an α-sequence constructed to meet α-many open dense sets. There is no problem if $\alpha' > \alpha$ since then $T_{\alpha'}$ is $\le \alpha$-closed. If $\alpha' = \alpha$ then the property follows from the definition of level $|p_\alpha|$ of T_α, since we can arrange that $\text{Even}(p_\alpha)$ is sufficiently generic for $T_\alpha \restriction$ (levels $< |p_\alpha|$). (In fact the latter genericity is a consequence of the usual construction of the α-sequence leading to p.)

The forcing $\mathscr{P}(\gamma)$, $\gamma < \kappa$, is designed to produce a real $R(\gamma)$ such that $n \in R(\gamma)$ iff $R(\gamma)$ codes a branch through T_α for each α in $\text{Card}(\gamma, n)$. A condition is $p \in \prod_n \mathscr{P}(\gamma, n)$ where $p(n)(0)$ (a finite object) is $(\varnothing, \varnothing)$ for all but finitely many n. Extension is defined by $q \le p$ iff $q(n) \le p(n)$ in $\mathscr{P}(\gamma, n)$ unless n is not of the form $2^{n_0}3^{n_1}$ or $n = 2^{n_0}3^{n_1}$ where $q(n_0)_0(n_1) = 0$, in which case there is no requirement on $q(n)$. A generic G can be identified with the real $\{2^n 3^m | p(n)_0(m) = 1 \text{ for some } p \in G\} = R(\gamma)$. The forcing at or above

α, $\mathscr{P}(\gamma)_\alpha$, obeys "quasi-distributivity": if D_i, $i < \alpha$, are predense below p then there are $q \leq p$ and $d_i \subseteq D_i$, $i < \alpha$, such that each d_i is countable and predense below q. This is established as in David [2] by "guessing at $\langle p(n)(0) | n \in \omega \rangle$" and yields cardinal preservation.

Our desired forcing $\mathscr{P}$ is the "diagonally supported" product of the $\mathscr{P}(\gamma)$, $\gamma < \kappa$. Specifically, a condition is $p \in \prod_{\gamma<\kappa} \mathscr{P}(\gamma)$ where for infinite cardinals $\alpha < \kappa$, $\{\gamma | p(\gamma)(\alpha) \neq (\varnothing, \varnothing)\}$ has cardinality $\leq \alpha$ and in addition $\{\gamma | p(\gamma)(0) \neq (\varnothing, \varnothing)\}$ is finite. Quasi-distributivity for $\mathscr{P}_\alpha$ = forcing at or above α follows just as for $\mathscr{P}(\gamma)_\alpha$. The point of the diagonal supports is that for infinite successor cardinals α, $\mathscr{P}$ factors as $\mathscr{P}_\alpha * \mathscr{P}^{G_\alpha}$ where G_α denotes the $\mathscr{P}_\alpha$-generic and $\mathscr{P}^{G_\alpha}$ is $\alpha^+ - CC$. Thus we get cardinal-preservation.

Now note that if $\langle R(\gamma) | \gamma < \kappa \rangle$ comes from (and therefore determines) a $\mathscr{P}$-generic then $n \in R(\gamma) \rightarrow R(\gamma)$ codes a branch through T_α for α in $\mathrm{Card}(\gamma, n)$. Conversely, if $n \notin R(\gamma)$ then there is no condition on extension of conditions in $\mathscr{P}(\gamma)$ to cause $R(\gamma)$ to code a branch through such T_α. In fact, by the quasi-distributivity argument for $\mathscr{P}_\alpha$, given any term τ for a subset of α^+ and any condition p, we can find $\beta < \alpha^+$ of cofinality α and $q \leq p$ such that q forces $\tau \cap \beta$ to be one of α-many possibilities, each constructed before β^*, where $\beta = |q_\alpha|$. Thus q forces that τ is *not* a branch through T_α, so we get: $n \in R(\gamma)$ iff $R(\gamma)$ codes a branch through each T_α, $\alpha \in \mathrm{Card}(\gamma, n)$, iff $R(\gamma)$ codes a branch through some T_α, $\alpha \in \mathrm{Card}(\gamma, n)$. The coding is localized in the sense that if $n \in R(\gamma)$ then whenever $L_\eta(R(\gamma)) \vDash ZF^- + \mathrm{Card} = \mathrm{Card}^L$, there is $\gamma^* < \kappa^*$ such that $L_\eta(R(\gamma)) \vDash R(\gamma)$ codes a branch through $T^*_{\alpha^*}$ whenever $\alpha^* \in \mathrm{Card}^*(\gamma^*, n)$, where κ^*, $T^*_{\alpha^*}$, $\mathrm{Card}^*(\gamma^*, n)$ are defined in L_η just as κ, T_α, $\mathrm{Card}(\gamma^*, n)$ are defined in L. The latter condition on $R(\gamma)$ is sufficient to know that $R(\gamma)$ is equal to one of the intended $R(\gamma)$, $\gamma < \kappa$, even if we restrict ourselves to countable η. With that restriction we get a Π^1_2 condition equivalent to membership in $X = \{R(\gamma) | \gamma < \kappa\}$. Since set-forcing preserves the Souslinness of trees at sufficiently large cardinals, the above Π^1_2 definition of X works in any set-generic extension of $L(X)$. This completes the proof of the Theorem.

Proof of Corollary 2. As in the proof of Corollary 1 we can obtain $X = \{R(\gamma) | \gamma < \kappa\}$, κ = 999th cardinal after the least L-inaccessible, which has a Π^1_2 definition uniform for set-generic extensions of $L(X)$, where $\mathrm{Card}^{L(X)} = \mathrm{Card}^L$. We can guarantee that $Y = \{\langle R(0), R(\gamma_1), R(\gamma_2)\rangle | 0 < \gamma_1 \leq \gamma_2 < \kappa\}$ also has such a uniform Π^1_2 definition, using the following trick: Design $R(0)$ so that $u \in R(0) \Leftrightarrow \mathrm{Even}(R(0))$ codes a branch through T_α for each α in $\mathrm{Card}(0, n)$, and so that $\mathrm{Odd}(R_0)$ almost disjointly codes $\{\langle R(\gamma_1), R(\gamma_2)\rangle | 0 < \gamma_1 \leq \gamma_2 < \kappa\}$. Thus, for $R \in L(X)$, R^* is almost disjoint from $\mathrm{Odd}(R_0)$ iff $R = \langle R(\gamma_1), R(\gamma_2)\rangle$ for some $0 < \gamma_1 \leq \gamma_2 < \kappa$, where $R^* = \{n | n$ codes a finite initial segment of $R\}$. The former requires only a very small modification to the definition of the $\mathscr{P}(0)$ forcings. The latter requires only a small modification to the definition of $\mathscr{P}$: take the diagonally-supported product as before, but restrain $p(0)$ for $p \in \mathscr{P}$ so as to affect the desired almost disjoint coding. These finite restraints do not interfere with the quasi-distributivity argument for $\mathscr{P}$.

Now we have the desired Π^1_2 definition for $Y = \{\langle R(0), R(\gamma_1), R(\gamma_2)\rangle | 0 < \gamma_1 \leq \gamma_2 < \kappa\}$: R belongs to Y iff $R = \langle R_0, R_1, R_2\rangle$ where $R_0 = R(0)$ and $\langle R_1, R_2\rangle^*$ is almost disjoint from R_0 and R_1, R_2 belong to X. Since $R(0)$

is uniformly definable as a Π^1_2-singleton in set-generic extensions of $L(X)$, this is the desired definition. Of course, using Y we obtain a Π^1_2 well-ordering of length κ. Finally, as in the proof of Corollary 1, gently collapse κ to ω_1 and we have $\omega_1 > \omega_1^{L(R)}$ for each real R with a Π^1_2 well-ordering of length $\aleph_{1000}$. $\square$

Remarks. The same proof gives length $\aleph_\alpha$ for any L-definable α. We can also add Cohen reals so that the continuum is as large as desired, without changing the maximum length of a Π^1_2 well-ordering.

It is possible to show that if $O^\#$ exists then there is a Π^1_2 set X such that X has large cardinality in $L(X)$. But this requires the more difficult technique of Friedman [3].

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

E-mail address: sdf@math.mit.edu