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i-CONVEXITY OF MANIFOLDS WITH REAL PROJECTIVE STRUCTURES

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ABSTRACT. We compare the notion of higher-dimensional convexity, as defined by Carrière, for real projective manifolds with the existence of hemispheres. We show that if an i-convex real projective manifold M of dimension n for an integer i with $0 < i < n$ has an i-dimensional hemisphere, then M is projectively homeomorphic to $\mathbf{S}^n/\Gamma$ where Γ is a finite subgroup of $O(n+1, \mathbf{R})$ acting freely on $\mathbf{S}^n$.

A real projective structure ($\mathbf{RP}^n$-structure) on a smooth manifold of dimension n, $n > 0$, is given by an atlas of charts to the sphere $\mathbf{S}^n$ where transition functions of the charts are restrictions of projective automorphisms of $\mathbf{S}^n$. An $\mathbf{RP}^n$-*manifold* is a manifold with an $\mathbf{RP}^n$-structure, and a *projective map* is an immersion-preserving real projective structures locally. The Klein model of hyperbolic geometry implies that n-dimensional hyperbolic manifolds provide examples of $\mathbf{RP}^n$-manifolds. (See [2] and [3].)

Throughout this paper, let M be an $\mathbf{RP}^n$-manifold; let $\widetilde{M}$ denote the universal cover of M with the induced $\mathbf{RP}^n$-structure. M has a developing map $\mathrm{dev}\colon \widetilde{M} \to \mathbf{S}^n$, a projective map. The sphere $\mathbf{S}^n$ has a standard Riemannian metric μ of curvature 1 and the associated distance metric $\mathbf{d}$. Note that dev induces a Riemannian metric μ on $\widetilde{M}$ from $\mathbf{S}^n$. Associated with μ is the induced distance metric $\mathbf{d}$ on $\widetilde{M}$. The completion $\check{M}$ of $\widetilde{M}$ is obtained by completing $\mathbf{d}$. Let σ be the frontier set $\check{M} - \widetilde{M}$. The sets $\check{M}$ and σ are topologically independent of the choice of dev, and dev extends uniquely to a distance decreasing map on $\check{M}$. The extended map is also called a developing map and is denoted by the same symbol dev.

Let i be an integer such that $0 < i \leq n$ holds. A *great i-sphere* is a totally geodesic i-dimensional sphere imbedded in $\mathbf{S}^n$; a subset of $\check{M}$ for which the restriction of dev is an imbedding onto a great i-sphere in $\mathbf{S}^n$ is also called a *great i-sphere.* A *great i-ball* is a hemisphere of a great i-sphere $\mathbf{S}^i$ in $\mathbf{S}^n$; a

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subset of $\check{M}$ for which the restriction of dev is an imbedding onto a great i-ball is also called a *great i-ball*. An n-dimensional open hemisphere of $\mathbf{S}^n$ has a natural affine structure. An *i-simplex* is the convex hull of $i+1$ independent points in the hemisphere (under affine geometry). An *i-simplex* in $\check{M}$ is a subset for which the restriction of dev is an imbedding onto an i-simplex in an n-dimensional open hemisphere of $\mathbf{S}^n$.

We introduce the definition given by Carrière [1]. We say that M is *i-convex* for an integer i with $0 < i < n$ if the following holds: Given an $(i+1)$-simplex T in $\check{M}$, if F_1 is a face of T such that $T \cap \sigma = F_1^o \cap \sigma$, then $T \subset \widetilde{M}$. (Note that the i-convexity of M implies the j-convexity of M where $i \leq j < n$.)

A subset A of $\check{M}$ is called *convex* if every two points of A are connected by an imbedded arc α such that $\mathrm{dev}|\alpha$ is an imbedding onto a segment of $\mathbf{d}$-length $\leq \pi$ (see [2, §1] for more detail). It is easy to see that given a convex open subset A of $\widetilde{M}$, the map $\mathrm{dev}|A$ is isometric with respect to the metrics on $\widetilde{M}$ and $\mathbf{S}^n$. Thus, $\mathrm{dev}|\mathrm{Cl}(A)$ is an imbedding onto $\mathrm{Cl}(\mathrm{dev}(A))$ where $\mathrm{Cl}(A)$ is the closure of A in $\check{M}$.

We prove the following theorem in this paper.

Theorem. *Suppose that M is i-convex where $0 < i < n$ holds. Suppose that $\widetilde{M}$ includes a great j-ball or a great j-sphere for $i \leq j < n$. Then $\widetilde{M}$ is projectively homeomorphic to $\mathbf{S}^n$.*

The conclusion implies that M is projectively homeomorphic to $\mathbf{S}^n/\Gamma'$ where Γ' is a subgroup of the projective automorphism group $\mathrm{Aut}(\mathbf{S}^n)$ acting freely and properly discontinuously on $\mathbf{S}^n$. It is a standard fact that Γ' is conjugate to a finite subgroup of $O(n+1, \mathbf{R})$, the group of isometries of $\mathbf{S}^n$. Thus, M is projectively homeomorphic to $\mathbf{S}^n/\Gamma$ where Γ is a finite subgroup of $O(n+1, \mathbf{R})$ acting freely on $\mathbf{S}^n$.

Proof of the Theorem. We give the proof of the theorem assuming that Lemma 1, which follows, holds. We can prove the theorem by using induction on j. Suppose that $j = n-1$. Then by Lemma 1 $\widetilde{M}$ includes a great sphere of dimension n. Therefore, $\widetilde{M}$ is projectively homeomorphic to $\mathbf{S}^n$.

Suppose that the conclusion is true for the case where $j = k > i$ holds. We verify the conclusion for the case where $j = k-1 \geq i$ holds. By Lemma 1, $\widetilde{M}$ includes a great k-sphere. By the induction hypothesis, $\widetilde{M}$ is projectively homeomorphic to $\mathbf{S}^n$. This completes the proof.

Let us discuss Lemma 1. For an integer j with $0 < j < n$, a $(j+1)$-*bihedron* in $\mathbf{S}^n$ is a closed domain in a great $(j+1)$-sphere $\mathbf{S}^{j+1}$ in $\mathbf{S}^n$ bounded by two great j-balls with common boundary equal to a great $(j-1)$-sphere or the set of two points antipodal to each other; a $(j+1)$-*bihedron* in $\check{M}$ is a subset for which the restriction of dev is an imbedding onto a $(j+1)$-bihedron in $\mathbf{S}^n$. The bounding great j-balls of a bihedron are called *faces*. A bihedron in $\mathbf{S}^n$ or $\widetilde{M}$ is convex if and only if the interior angle between two faces of the bihedron is less than or equal to π.

Lemma 1. *Assume that M is i-convex for an integer i with $0 < i < n$. Suppose that $\widetilde{M}$ includes a great j-ball B_0 where $i \leq j < n$. Then $\widetilde{M}$ includes a great $(j+1)$-sphere.*

Proof. We choose a convex $(j+1)$-bihedron T_0 including B_0 in a neighborhood of B_0 with faces B_a and B_b such that $\delta B_a = \delta B_b = \delta B_0$ holds. (Assume that B_0 is not a face of T_0.) Let us agree that bihedra in this proof are $(j+1)$-dimensional always. Let A^+ be the set of convex bihedra with a face B_0 and including B_a, and let A^- be the set of convex bihedra with a face B_0 and including B_b. Then there is a unique great $(j+1)$-sphere $\mathbf{S}^{j+1}$ in $\mathbf{S}^n$ including the images of elements of A^+ and A^- under dev.

We may parameterize A^+ and A^- by positive intervals. Given an element T of A^+ or A^-, let $\theta(T)$ be the interior angle between B_0 and the other face of T. This defines a function θ from the set of elements of A^+ and A^- to $\mathbf{R}$. Let T_a be the bihedron in T_0 bounded by B_0 and B_a; let T_b be the bihedron in T_0 bounded by B_0 and B_b. Suppose that T' and T'' are two bihedra in A^+ with $\theta(T') = \theta(T'')$. Then Lemma 2 implies that $T' = T''$. Thus $\theta|A^+$ is an injective map into $[\theta(T_a), \pi]$. Similarly, $\theta|A^-$ is an injective map into $[\theta(T_b), \pi]$.

If we have $t < t'$ where $t' \in \theta(A^+)$ and $t \in [\theta(T_a), \pi]$ hold, then t is realized as the angle of a bihedron in A^+ which is included in the bihedron corresponding to t'. It follows that $\theta(A^+)$ is connected. Similarly, $\theta(A^-)$ is connected.

Let $T \in A^+$. Then $\operatorname{dev}|T$ is an imbedding onto a convex bihedron $\operatorname{dev}(T)$. Choose an open neighborhood N of T in $\widetilde{M}$ such that $\operatorname{dev}|N$ is an imbedding onto an open subset of $\mathbf{S}^n$. Then for every convex bihedron T' in $\operatorname{dev}(N)$ the open subset N includes a convex bihedron T'' such that $\operatorname{dev}(T'') = T'$. This implies that $[\theta(T_a), \pi]$ includes an open neighborhood of $\theta(T)$ whose elements are realized by bihedra in A^+. Hence, $\theta(A^+)$ is an open subset of $[\theta(T_a), \pi]$. Similarly, $\theta(A^-)$ is an open subset of $[\theta(T_b), \pi]$.

We claim that $\theta(A^+)$ is closed. (We use i-convexity now.) Suppose that it is not closed. Then $\theta(A^+)$ is the half-open interval $[\theta(T_a), t^+)$ for a real number t^+ less than or equal to π. Let $T_1 = \bigcup_{T \in A^+} T^o$. Since by Lemma 2, $\operatorname{dev}|T_1$ is injective, $\operatorname{dev}|T_1$ is an imbedding onto the interior of a convex bihedron of angle t^+. For the closure $\mathrm{Cl}(T_1)$ of T_1 in $\check{M}$, the map $\operatorname{dev}|\mathrm{Cl}(T_1)$ is an imbedding onto $\mathrm{Cl}(\operatorname{dev}(T_1))$. Since $\mathrm{Cl}(\operatorname{dev}(T_1))$ is a convex bihedron, so is $\mathrm{Cl}(T_1)$. The bihedron $\mathrm{Cl}(T_1)$ has two faces B_0 and $B_{\mathrm{Cl}(T_1)}$ where $\sigma \cap \mathrm{Cl}(T_1) \subset B^o_{\mathrm{Cl}(T_1)}$. Since $\sigma \cap B_{\mathrm{Cl}(T_1)}$ is compact, $\sigma \cap B^o_{\mathrm{Cl}(T_1)}$ is a bounded subset of the open great ball $B^o_{\mathrm{Cl}(T_1)}$. Thus, the bihedron $\mathrm{Cl}(T_1)$ includes a $(j+1)$-simplex K such that $\sigma \cap \mathrm{Cl}(T_1) \subset K^o_1$ where K_1 is a j-dimensional face of K and is included in $B_{\mathrm{Cl}(T_1)}$. The definition of i-convexity implies that $K_1 \cap \sigma = \varnothing$ and $\mathrm{Cl}(T_1) \subset \widetilde{M}$ hold. Since $\mathrm{Cl}(T_1) \supset T_a$, we have $\mathrm{Cl}(T_1) \in A^+$ and hence $t^+ \in \theta(A^+)$. This is absurd. Therefore, $\theta(A^+)$ and, similarly, $\theta(A^-)$ are closed.

We therefore have $\theta(A^+) = [\theta(T_a), \pi]$ and $\theta(A^-) = [\theta(T_b), \pi]$. Let T^+ be the convex bihedron corresponding to π belonging to A^+, and let T^- be that belonging to A^-. We have $T^+, T^- \subset \widetilde{M}$. The maps $\operatorname{dev}|T^+$ and $\operatorname{dev}|T^-$ are imbeddings onto great $(j+1)$-balls in the great $(j+1)$-sphere $\mathbf{S}^{j+1}$. Since the intersection $\operatorname{dev}(T^+) \cap \operatorname{dev}(T^-)$ is a great j-sphere and, hence, is path-connected, Lemma 2 implies that $\operatorname{dev}|T^+ \cup T^-$ is an imbedding onto $\mathbf{S}^{j+1}$. Therefore, $T^+ \cup T^-$ is a great $(j+1)$-sphere in $\widetilde{M}$. This completes the proof of Lemma 1.

Lemma 2. *Suppose that* A *and* B *are path-connected compact subsets of* $\widetilde{M}$ *such that* $\operatorname{dev}|A$ *and* $\operatorname{dev}|B$ *are imbeddings. Suppose that* $A \cap B \neq \varnothing$ *and that* $\operatorname{dev}(A) \cap \operatorname{dev}(B)$ *is a path-connected subset of* $\mathbf{S}^n$.

Then $\operatorname{dev}|A \cup B$ *is an imbedding onto* $\operatorname{dev}(A) \cup \operatorname{dev}(B)$.

Proof. We only need to deduce the injectivity of $\operatorname{dev}|A \cup B$ from the fact that given a path in $\mathbf{S}^n$ and an initial point in $\widetilde{M}$ there is at most one lift of the path to $\widetilde{M}$ (see [1, Proposition 1.3.1]).

Let us end this paper with the following remark: Carrière conjectured in 1988 that the homotopy groups in dimensions greater than or equal to i, $i > 1$, for an i-convex affine manifold are trivial. This conjecture is still open. What we did here may aid us in understanding similar questions for projective manifolds.

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