

THE MONOTONE PRINCIPLE OF FIXED POINTS: A CORRECTION

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ABSTRACT. Some technical aspects of the main result in Taskovic's *A monotone principle of fixed points* [Proc. Amer. Math. Soc. 94 (1985), 427–432] are investigated.

In the 1985 paper by Taskovic [3], a monotone principle of fixed points is established, which is claimed to include those of Boyd and Wong [1] or Dugundji and Granas [2]. The core of this result is the following auxiliary fact. Let F stand for the class of all functions $\gamma: [0, \infty) \rightarrow [0, \infty)$, with

$$(D_1) \quad \gamma(0) = 0 \quad \text{and} \quad \gamma(t) < t, \quad \text{for all } t > 0.$$

Lemma. Suppose $\gamma \in F$ satisfies

$$(c) \quad \limsup_{s \rightarrow t_+} \gamma(s) < t \quad \text{whenever } t > 0.$$

Then each double indexed sequence $(t_{n,m})$ in $[0, \infty)$ which is bounded from above and fulfills

$$(1) \quad t_{n+1,m+1} \leq \gamma(t_{n,m}), \quad n, m = 0, 1, \dots,$$

must converge to zero as $n, m \rightarrow \infty$.

We shall prove by an example that (c) is not sufficient for such a conclusion (hence for the validity of the quoted fixed point principle). So the question arises of what must be added for the above conclusion to hold. It is our aim in this note to identify this condition; some other aspects occasioned by these developments are also discussed.

To give the promised example, we let $\gamma \in F$ be defined as

$$(D_2) \quad \gamma(t) = \begin{cases} 0 & \text{if } t \in [0, 1/2] \cup [1, \infty), \\ 2t - 1 & \text{if } t \in (1/2, 1). \end{cases}$$

Clearly, (c) holds for this function. Construct a double indexed sequence $(t_{n,m})$ in $[0, \infty)$ by

$$(D_3) \quad t_{n,m} = \begin{cases} 0 & \text{if } 2n \geq m, \\ 1 - 2^{-(m-2n)} & \text{otherwise.} \end{cases}$$

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That $(t_{n,m})$ is bounded above and fulfills (1) is clear. We also have

$$(2) \quad t_{k,3k} = 1 - 2^{-k} \rightarrow 1 \quad \text{as } k \rightarrow \infty.$$

This, however, cannot be in agreement with

$$t_{n,m} \rightarrow 0 \quad \text{as } n, m \text{ tend to infinity.}$$

Hence, the lemma is false, as claimed.

Roughly speaking, the “bad” convergence character of the sequence in the statement is due to the condition

$$(b) \quad \limsup_{s \rightarrow t_-} \gamma(s) = t \quad \text{for some } t > 0$$

fulfilled by the considered function (in $t = 1$). For, if this happens, then, in combination with (D_1) , a strictly ascending sequence (s_n) in $[0, t]$ may be constructed with $s_0 = 0$ and

$$(3) \quad s_n < \gamma(s_{n+1}) < s_{n+1}, \quad n = 0, 1, \dots$$

As a consequence, the double indexed sequence

$$(D_4) \quad t_{n,m} = \begin{cases} s_{n-2n} & \text{if } 2n < m, \\ s_0 & \text{otherwise} \end{cases}$$

fulfills (1); and this, coupled with

$$(4) \quad t_{k,2k+1} = s_1 \neq 0, \quad k = 1, 2, \dots,$$

shows the lemma cannot (in general) hold. So, in order that its conclusion be retainable, a condition like

$$(b^*) \quad \limsup_{s \rightarrow t_-} \gamma(s) < t, \quad t > 0,$$

is indispensable. This, in the context of (c), is equivalent with $\gamma \in F$ being majorized by an increasing function $\delta \in F$, which also fulfills (c); for instance,

$$\delta(t) = \sup\{\gamma(s); 0 \leq s \leq t\}, \quad t \geq 0.$$

Hence (relabeling this new function) we may assume in the statement that, in addition to the accepted conditions,

$$(d) \quad \gamma \text{ is increasing over } [0, \infty).$$

Note that, as a consequence of these,

$$(e) \quad \gamma^n(t) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \text{for all } t \geq 0.$$

(Here, γ^n stands for the n th iterate of γ .) We now claim that, in the context of (d), the obtained property will suffice for the validity of the lemma. Indeed, let τ be any upper bound of $(t_{n,m})$. We have, for any couple n, m of positive integers

$$t_{n,m} \leq \gamma^k(\tau), \quad \text{where } k = \min(n, m).$$

Using (e), the conclusion is clear.

It follows from this that, in the increasing context, the most natural strategy in the lemma is to adopt (e) in place of (c). To see this is an effective extension of the preceding one, denote

$$M(\gamma) = \{t > 0; \limsup_{s \rightarrow t_+} \gamma(s) = t\}.$$

Then, even if $M(\gamma)$ is denumerable around the origin, (e) may eventually hold; see Turinici [4] for details. An open problem is to what extent can

(a) $M(\gamma)$ is nondenumerable around the origin

be in agreement with (e); we conjecture that the answer is negative.

As a consequence of these, the monotone principle of fixed points in Taskovic [3] holds if, in addition to (c), the contractivity function $\gamma \in F$ introduced by its statement also fulfills (b*). This has, in general, a negative impact on the methodological possibilities of deriving from it—in the way described there—the fixed point results due to Boyd and Wong [1] or Dugundji and Granas [2]. The question of whether or not this is possible via different procedures remains open.

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