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ON THE DIMENSION OF CERTAIN TOTALLY DISCONNECTED SPACES

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ABSTRACT. It is well known that there exist separable, metrizable, totally disconnected spaces of all dimensions. In this note we introduce the notion of an almost 0-dimensional space and prove that every such space is a totally disconnected subspace of an **R**-tree and, hence, at most 1-dimensional. As applications we prove that the spaces of homeomorphisms of the universal Menger continua are 1-dimensional and that hereditarily locally connected spaces have dimension at most two.

1. INTRODUCTION

All spaces are separable and metric. Let X be a space. The *quasicomponent* of a point $x \in X$ is the intersection of all subsets of X which are both open and closed and which contain x. A space is *totally disconnected* provided each quasicomponent is a point. Compact, Hausdorff, totally disconnected spaces are 0-dimensional. However, it is well known that there exist even complete, separable, totally disconnected, metric spaces of all dimensions.

A space X is 0-dimensional (resp., *almost 0-dimensional*) provided there exists a basis $\mathscr{B}$ for X such that for each $B \in \mathscr{B}$, $X \backslash \overline{B}$ is a clopen set (resp., $X \backslash \overline{B}$ is a union of clopen sets). Note that each 0-dimensional space is almost 0-dimensional and that each almost 0-dimensional space is totally disconnected. The set of points in Hilbert space, all of whose coordinates are irrational, is an example of a 1-dimensional, almost 0-dimensional, totally disconnected space.

We prove that almost 0-dimensional separable metric spaces are embeddable in **R**-trees and, hence, have dimension bounded above by one. We will give two applications of this result; other applications will appear in a later note.

First we show that the spaces of homeomorphisms of the universal Menger continua are 1-dimensional. This extends a well-known result of Brechner [4], who had shown that the dimension of these spaces is at least one. Next we provide a partial answer to a question of R. Duda (1964) by proving that the

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dimension of a hereditarily locally connected, separable, metric space does not exceed two.

Let X be a metric space with a given metric d. We will denote by $B(x, \varepsilon)$ the open ball of radius ε around x and by $\widehat{B}(x, \varepsilon)$ the closed ball of radius ε around x. We will call a set which is both open and closed, clopen. For basic results and definitions which are not given here the reader may consult Kuratowski [6] or Whyburn [15].

2. Main theorem

A space is said to be *arcwise connected* if each pair of its distinct points is contained in an arc in the space (i.e., a subset of the space which is homeomorphic to the closed unit interval). A space is *uniquely arcwise connected* if it is arcwise connected and if it does not contain a homeomorphic copy of the unit circle in the plane.

A space X is said to be an $\mathbf{R}$-*tree* if it is connected, locally arcwise connected, and uniquely arcwise connected. It is known (see [10]) that a nondegenerate $\mathbf{R}$-tree has dimension one. Every connected subset of an $\mathbf{R}$-tree is an $\mathbf{R}$-tree. A compact $\mathbf{R}$-tree is called a dendrite. A point $x \in X$ is said to be an *end-point* of X if x is an end-point of each arc in X which contains x. It is easy to see (see [10]) that the set of end-points of an $\mathbf{R}$-tree is totally disconnected.

We shall prove that a space X is almost 0-dimensional if and only if X is homeomorphic to the set of end-points of some $\mathbf{R}$-tree.

Lemma 1. *Let X be almost 0-dimensional with respect to the basis $\mathscr{B}$. Then for each $B, C \in \mathscr{B}$ such that $\overline{B} \cap \overline{C} = \varnothing$, there exist disjoint clopen sets H and K in X such that $X = H \cup K$, $B \subset H$, and $C \subset K$.*

Proof. Suppose X is almost 0-dimensional with respect to the basis $\mathscr{B}$ and $B, C \in \mathscr{B}$ have disjoint closures. Then $X\backslash\overline{B} = \bigcup H_n$ and $X\backslash\overline{C} = \bigcup K_m$, where $\{H_n\}$ and $\{K_m\}$ are families of pairwise disjoint clopen sets. Consider

$$G_n = H_n \backslash \bigcup\{K_1, \dots, K_n\},$$

and let $F = \bigcup G_n$. Then each G_n is clopen, $C \subset F$, and $F \cap B = \varnothing$. It remains to be shown that F is clopen. Since F is a union of open sets, it is clearly open. Hence suppose $x \in \overline{F}$. Suppose first that $x \in \overline{C}$. Then $x \notin K_m$ for all m and $x \in H_n$ for some n. Hence, $x \in G_n \subset F$. Assume next that $x \notin \overline{C}$. Then $x \in K_m$ for some m. Since K_m is an open neighborhood of x and $G_n \cap K_m = \varnothing$ for all $n \geq m$, $x \in \overline{\bigcup_{n<m} G_n}$. Since each G_n is closed, $x \in G_n$ for some $n < m$ and $x \in F$. This completes the proof of the lemma.

Theorem 2. *Let X be an almost 0-dimensional space. Then X embeds in (the set of end-points of) an $\mathbf{R}$-tree. In particular, X is at most 1-dimensional.*

Proof. Let X be almost 0-dimensional with respect to the countable basis $\mathscr{B} = \{B_i\}$. We will assume that X is a subset of $Q \times \{0\} \subset Q \times I$, where Q is the Hilbert cube with its usual metric d and I is the interval $[0, 1]$.

The main idea of the proof is as follows: We will construct a uniquely arc-connected subset R of $Q \times I$ such that $R \cap \pi^{-1}(0) = X$, where $\pi : Q \times I \to I$ is the natural projection. The space R is then given a finer topology which makes it an $\mathbf{R}$-tree with X as a closed set of end-points.

Let $\mathscr{U}_1 = \{X\}$, $x_{X,1} \in B_1$, $X_1 = \{x_{X,1}\}$; and let T_1 be the arc $\{r\} \times I$ where $x_{X,1} = (r, 0)$. Then $x_{X,1}$ is an end-point of T_1.

Suppose $\mathscr{U}_n \prec \mathscr{U}_{n-1} \prec \cdots \prec \mathscr{U}_1$ are finite covers of X of minimum cardinality by pairwise disjoint clopen sets such that if $1 \le r < s \le i \le n$ and $\overline{B}_r \cap \overline{B}_s = \varnothing$, then no element of $\mathscr{U}_i$ meets both B_r and B_s. Hence, each element $U \in \mathscr{U}_n$ meets at least one of $\{B_1, \dots, B_n\}$. Suppose $X_1 \subset X_2 \subset \cdots \subset X_n$ are subsets of X such that for each $i \le n$, X_i meets each $U \in \mathscr{U}_i$ in exactly one point $x_{U,i}$ and if $x_{U,i} \in X_i \backslash X_{i-1}$, then $x_{U,i} \in B_j$ for some $j \le i$ such that $\operatorname{diam}(B_j)$ is minimal among $\{B_k \mid k \le i \text{ and } B_k \cap U \ne \varnothing\}$. Suppose $T_1 \subset T_2 \subset \cdots \subset T_n \subset Q \times I$ are finite trees such that for $i \le n$, X_i is a set of end-points of T_i and $T_i \cap \pi^{-1}(0) = X_i$. For $U \in \mathscr{U}_i$ let $K_{U,i}$ be a free arc in T_i (i.e., $K_{U,i}$ minus its end-points is an open set in T_i) with end-point $x_{U,i}$ and so $K_{U,i} \cap K_{V,i} = \phi$ for $U \ne V$ in $\mathscr{U}_i$.

Suppose $\mathscr{U}_n$, T_n, and X_n have been constructed. By Lemma 1, there is a finite cover $\mathscr{U}_{n+1}$ of X of minimal cardinality of pairwise disjoint open sets in X such that $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$ and such that if $1 \le r < n+1$ and $\overline{B_r} \cap \overline{B_{n+1}} = \varnothing$, then no element of $\mathscr{U}_{n+1}$ meets both B_r and B_{n+1}. If $U \in \mathscr{U}_{n+1}$ and $U \cap X_n = \varnothing$, let $x_{U,n+1} \in B_i \cap U$ where $1 \le i \le n+1$ and B_i has minimal diameter among $\{B_j \mid j \le n+1 \text{ and } B_j \cap U \ne \varnothing\}$. If $U \in \mathscr{U}_{n+1}$ and $U \cap X_n \ne \varnothing$, let $W \in \mathscr{U}_n$ such that $U \subset W$ and let $x_{U,n+1} = x_{W,n}$. Let $X_{n+1} = \{x_{U,n+1} \mid U \in \mathscr{U}_{n+1}\}$.

Let $U \in \mathscr{U}_{n+1}$ and $W \in \mathscr{U}_n$ such that $U \subset W$. If $U \cap X_n \ne \varnothing$, let $A_{U,n+1} = \varnothing$ and if $U \cap X_n = \varnothing$, let $A_{U,n+1}$ be an irreducible piecewise linear arc from $x_{U,n+1}$ to T_n such that $\pi \mid A_{U,n+1}$ is a homeomorphism, $A_{U,n+1}$ meets $K_{W,n}$, $A_{U,n+1} \cap A_{V,n+1} = \phi$ for $U \ne V$ in $\mathscr{U}_{n+1}$, $A_{U,n+1}$ is contained in the 2^{-n-1} ball around the straight line segment $[x_{U,n+1}, x_{W,n}]$ in $Q \times I$ and the length of the arc $A_{U,n+1}$ is less than $d(x_{U,n+1}, x_{W,n}) + 2^{-n-1}$. Let

$$T_{n+1} = T_n \cup \bigcup \{A_{U,n+1} \mid U \in \mathscr{U}_{n+1}\}.$$

By induction $\mathscr{U}_n$, T_n, and X_n are constructed for each positive integer n. For $y, z \in \bigcup T_n$ there is an integer m such that $y, z \in T_m$. Let $[y, z]$ be the unique arc in T_m from y to z. Then $[y, z]$ is well defined and $X \cap [y, z] \subset \{y, z\}$. Note that $\bigcup T_n \backslash X$ is a locally finite simplicial complex which is a tree.

Let $R = X \cup (\bigcup T_n)$. We will first consider R with the subspace topology and prove that in this topology R is uniquely arcwise connected. Next we change the metric (and, hence, the topology) on R such that it becomes locally arc connected. Since this new topology coincides with the subspace topology on arcs and on X, we will have completed the argument.

Choose a point $x \in X$. We will first show that there exists an arc in R joining x to $x_{X,1}$. Next we will indicate how to modify the argument to show that given $\varepsilon > 0$, there exists a neighborhood B of x in X such that for each $y \in B$, there exists an arc A in R joining x to y such that $\operatorname{diam}(A) < \varepsilon$.

For each n there exists a unique $U_n \in \mathscr{U}_n$ such that $x \in U_n$. Put $z_n = x_{U_n,n}$. It suffices to show that $\lim z_n = x$ since in that case the subtree $T(x)$ spanned by $\{x\} \cup \bigcup \{z_n\}$ is a dendrite and, hence, a locally connected continuum. Each pair of points of $T(x)$ is contained in an arc in $T(x)$ which meets X only in its end-points. Let $\{B_{i_n}\}$ be a sequence of basic sets containing x

such that $\operatorname{diam}(B_{i_n}) \le 1/n$, $B_{i_1} = X$, $\overline{B}_{i_n} \subset B_{i_{n-1}}$, and $i_{n-1} < i_n$.

It suffices to show that for each n there exists an integer a_n such that $z_p \in B_{i_n}$ for each $p \ge a_n$. The proof is by induction on n. Let $a_1 = 1$, and suppose a_{n-1} has been defined. Let $m > n$ be a positive integer such that $B(x, \frac{2}{m}) \subset B_{i_n}$. If $z_q \in B_{i_n}$ for each $q \ge i_m + a_{n-1}$ let $a_n = i_m + a_{n-1}$. Otherwise, let $q \ge a_{n-1} + i_m$ be smallest such that $z_q \notin B_{i_n}$. Since $\mathscr{B}$ is a basis for X, there is an integer $j \ge q$, such that $z_q \in B_j$, and $\overline{B}_{i_m} \cap \overline{B}_j = \phi$. Then $z_q \notin U_j$. Let a_n be the smallest integer such that $z_q \notin U_{a_n}$. Then $q < a_n \le j$. Then $z_{a_n - 1} = z_q$. For $p \ge a_n$ $z_p \in B_s$ for some $s \in \{1, \dots, p\}$ such that $\overline{B_s} \cap \overline{B}_{i_m} \ne \phi$ and $\operatorname{diam} B_s \le \operatorname{diam} B_{i_m} \le \frac{1}{m}$ since $x \in B_{i_m} \cap U_p$. It follows that $z_p \in B_{i_n}$.

Let $U = \bigcap \{B_j \mid j \le a_n \text{ and } x \in B_j\}$. Let $y \in U \cap U_{a_n}$. For each positive integer r let V_r be the member of $\mathscr{U}_r$ such that $y \in V_r$ and let $\{y_r\} = V_r \cap T_r$. Then for $p \le a_n$, $V_p = U_p$ and $y_p = z_p$. For $p \ge a_n$, $y_p \in B_{i_n}$ as above. The arc $[y, y_{a_n}] = [y, z_{a_n}]$ has diameter less than $\frac{2}{n} + 2^{-a_n+1}$ since it lies by construction very close to the ball $B(x, \frac{1}{n})$ in Q. It follows that each pair of points y and z in $U \cap U_{a_n}$ is contained in an arc $[y, z] \subset (T(y) \cup T(z) \backslash X) \cup \{y, z\}$ of diameter $< \frac{2}{n} + 2^{-a_n+2}$.

Let $x, y \in X$ with $x \ne y$. Let k be the smallest integer such that x and y lie in disjoint members U and V, respectively, of $\mathscr{U}_k$. Each point z of the nonempty set $[x_{U,k}, x_{V,k}] \backslash (K_{U,k} \cup K_{V,k})$ separates $X \cup T_k$ between U and V. By induction z separates $X \cup T_{k+j}$ between U and V for each positive integer j. It follows that each pair of points x and y of R is contained in a unique arc $[x, y]$ in $(R \backslash X) \cup \{x, y\}$.

Consider the following metric[1] on R:

$$\rho(x, y) = \operatorname{diam}([x, y]) \quad \text{for } x, y \in R.$$

Then ρ is clearly a metric which induces the same topology as the subspace topology on all arcs in R and on the subspace $R \backslash X$. Hence R, in the topology induced by ρ, is still uniquely arcwise connected. By definition it is locally arcwise connected in this topology and, hence, an **R**-tree. It remains to be shown that the topology induced on X by ρ coincides with the original topology (induced by the metric d) on X. Since $d(x, y) < \rho(x, y)$ for all $x, y \in X$, for each sequence $x_n \to x$ in the ρ metric $x_n \to x$ in the d metric. Hence assume $d(x_n, x) \to 0$. Let $\varepsilon > 0$. By the above argument there exists a neighborhood B of x such that for each $y \in B$, there exists an arc of diameter $< \varepsilon$ joining x to y. Hence $\rho(x_n, x) < \varepsilon$ for each $x_n \in B$. This completes the proof of the theorem.

3. Applications

Spaces of homeomorphisms. For each pair of positive integers n and k, $n < k$, Menger [11] has described an n-dimensional continuum M_k^n which is universal with respect to containing homeomorphic copies of every n-dimensional compactum which can be embedded in E^k. For any $k > 0$, M_k^0 is the Cantor set and M_2^1 is the Sierpiński universal plane continuum, and for $n \ge 1$, M_{2n+1}^n

[1] Introduction of this metric simplified the original argument. We are indebted to John Mayer for this suggestion.

is the n-dimensional Menger universal continuum. Alternative constructions and related results are given in [3, 5, 7, 14]. Anderson [1] for $n = 1$ and Bestvina [2] have provided elegant characterizations of the Menger universal continua and proved their homogeneity. Cannon [5] and Whyburn [15] characterized the $(n-1)$-dimensional Sierpiński Continua M_n^{n-1}, $n \neq 4$; and Lewis announced that M_k^n is not homogeneous for $n > 0$ and $k < 2n+1$. Let $\mathscr{H}_k^n$ denote the space of homeomorphisms of M_k^n with the sup metric d. Brechner [4] has essentially shown that $\mathscr{H}_{2n+1}^n$ is at least 1-dimensional and totally disconnected. It was conjectured that this space is infinite dimensional. We will prove that it is almost 0-dimensional and, hence, 1-dimensional. We first establish the following lemma.

Lemma 3. *Let X be a metric space, $x \in X$, and let*

$$N = \{\varepsilon > 0 \mid \overline{B(x, \varepsilon)} \neq \widehat{B}(x, \varepsilon)\}.$$

Then N is at most countable.

Proof. Let $\mathscr{B} = \{B_i\}$ be a countable basis for X. Note that for each $\varepsilon \in N$, there exists a point $x_\varepsilon \in \widehat{B}(x, \varepsilon)$ and a number $\eta_\varepsilon > 0$ such that

$$B(x, \varepsilon) \cap B(x_\varepsilon, \eta_\varepsilon) = \varnothing. \tag{1}$$

For each $\varepsilon \in N$, choose a basic set $B_{i_\varepsilon} \in \mathscr{B}$ such that $x_\varepsilon \in B_{i_\varepsilon} \subset B(x_\varepsilon, \eta_\varepsilon)$. If N is uncountable, then there exists an index n and $\varepsilon_1 < \varepsilon_2$ in N such that $i_{\varepsilon_1} = i_{\varepsilon_2} = n$. Then $x_{\varepsilon_1} \in B_n \subset B(x_{\varepsilon_2}, \eta_{\varepsilon_2})$. This contradicts (1) and completes the proof of the lemma.

Theorem 4. *Let X be a metric space and D a dense set of points of X. Suppose that for each $d \in D$ there exists a dense G_δ set E_d of positive numbers such that for each $\varepsilon \in E_d$, $X \backslash \widehat{B}(d, \varepsilon)$ is a union of clopen sets. Then X is almost 0-dimensional.*

Proof. By Lemma 3, for each $d \in D$, there exists a dense G_δ, $F_d \subset E_d$, such that $\overline{B(d, \varepsilon)} = \widehat{B}(d, \varepsilon)$ for each $\varepsilon \in F_d$. Hence, the result follows immediately.

Theorem 5. *$\mathscr{H}_k^n$ is almost 0-dimensional (and, hence, at most 1-dimensional) for $k \geq n+1$.*

Proof. By Theorem 4, it suffices to show that the complement of each closed ball in $\mathscr{H}_k^n$ is a union of clopen sets. Hence fix $\mathscr{H}_k^n$ $(k \geq n+1)$, $g \in \mathscr{H}_k^n$, and $\varepsilon > 0$. We shall show that for each $h \in \mathscr{H}_k^n$ with $d(h, g) > \varepsilon$, there exists a clopen set U in $\mathscr{H}_k^n$ containing h such that $U \cap \widehat{B}(g, \varepsilon) = \varnothing$. Suppose $d(h, g) = \varepsilon + 4\delta$, $\delta > 0$, and $x \in M_k^n$ such that $d(h, g) = d(h(x), g(x))$. Choose a n-sphere S in M_k^n such that $g(S) \subset B(g(x), \delta)$ and $h(S) \subset B(h(x), \delta)$. By Nagata [12], there exists a retraction $r: M_k^n \to h(S)$ such that $r(M_k^n \backslash B(h(x), 2\delta)) = \{\text{point}\}$. Let $U = \{f \in \mathscr{H}_k^n \mid r \circ f|S \not\simeq *\}$. Then U is clopen in $\mathscr{H}_k^n$ since near maps into an ANR are homotopic. Also $h \in U$ and for each $f \in B(g, \varepsilon)$, $f(S) \subset B(g(x), \varepsilon+\delta)$, so $r \circ f(S)$ is a point and $f \notin U$.

Corollary 6. *The set of homeomorphisms $\mathscr{H}_k^s$ of M_k^s is 1-dimensional for $(s, k) \in \{(n, 2n+1) \mid n \geq 1\} \cup \{(n, n+1) \mid 1 \leq n \neq 3\}$.*

Proof. A space X is locally setwise homogeneous if there exists a basis $\mathscr{B}$ of connected open sets and a dense countable subset $D \subset X$ such that for any $B \in \mathscr{B}$ and any two points $x, y \in B \cap D$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$ and $h | X \backslash B = \mathrm{id}_{X \backslash B}$. Brechner [4] has shown that the group of homeomorphisms of any locally setwise homogeneous continuum is at least 1-dimensional. Hence (cf. [4, 5, 2]), $\mathscr{H}_k^s$ is at least 1-dimensional. By Theorems 5 and 2, $\mathscr{H}_k^s$ is at most 1-dimensional and the result follows.

Hereditarily locally connected spaces. A space is said to be *hereditarily locally connected* if it is connected and if each of its connected subsets is locally connected. Clearly, each **R**-tree is hereditarily locally connected. We shall prove that each hereditarily locally connected space X can be written as $X = Y \cup C$ where Y is almost 0-dimensional and C is countable. It follows that the dimension of each hereditarily locally connected space does not exceed two. We recall two results from [13, p. 586 and p. 597] concerning hereditarily locally connected spaces.

Theorem 7. *If X is a hereditarily locally connected space and Y is a subset of X, then the components of Y are also the quasicomponents of Y.*

Theorem 8. *Let X be a hereditarily locally connected space, T a totally disconnected subset of X, and Y a closed subset of X. Then the nondegenerate components of $Y \cup T$ are the nondegenerate components of Y.*

The following corollary is of independent interest.

Corollary 9. *If Y is a totally disconnected subset of a hereditarily locally connected space X, then Y is almost 0-dimensional.*

Proof. Let $\{V_i\}_{i=1}^{\infty}$ be a basis of connected open sets for X. For each positive integer i let $U_i = V_i \cap Y$. Suppose $\overline{V}_i \cap \overline{V}_j = \varnothing$. By Theorem 8, $\overline{V}_i$ and $\overline{V}_j$ are components of $Y \cup \overline{V_i \cup V_j}$. Hence, Y can be separated between U_i and U_j by Theorem 7.

It is known [13, p. 573] that there is an **R**-tree with a 1-dimensional set E of end-points. By Corollary 9, E is almost 0-dimensional. The set of irrational sequences in the Hilbert space l_2 and the set of end-points of the universal separable **R**-tree [8] are other examples. (By a recent announcement of Fokkink and Oversteegen, see [9], these spaces are in fact homeomorphic.)

The following corollary follows immediately from Corollary 9 and Theorem 2.

Corollary 10. *A space X is almost 0-dimensional if and only if it can be embedded in the set of end-points of an* **R**-*tree.*

It is a classical result of Whyburn (see [15, p. 94]) that a compact hereditarily locally connected space X has a basis of open sets with countable boundaries and, hence, has dimension one. In [13, Theorem 2.4] Whyburn's theorem was improved by replacing compactness of X by semicolocal connectedness of X. (A space X is *semicolocally connected* if X has a basis of open sets $\{V_i\}_{i=1}^{\infty}$ such that for each i $X \backslash V_i$ has finitely many components.) In [13, p. 593] it was shown (by constructing an **R**-tree T with a set of end-points which is *not* rim-countable) that semicolocal connectedness is necessary in that result. R. Duda posed the following (see [13, §5]):

Problem 1. If X is a separable, metric, nondegenerate, hereditarily locally connected space, is $\dim(X) = 1$?

The following theorem provides a partial affirmative solution to Duda's problem.

Theorem 11. *If X is a separable, metric, hereditarily locally connected space, then $X = Y \cup C$ where Y is almost 0-dimensional and C is countable. In particular,* $\dim(X) \leq 2$.

Proof. By [13, Theorem 2.3], $X = Y \cup C$ where Y is a totally disconnected set and C is a countable set. By Corollary 9, Y is almost 0-dimensional. By Theorem 2, $\dim(Y) \leq 1$ and, hence,

$$\dim(X) \leq \dim(Y) + \dim(C) + 1 \leq 1 + 0 + 1 = 2.$$

References

1. R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. (2) **67** (1958), 313–324.
2. Mladen Bestvina, *Characterizing k-dimensional universal Menger compacta*, Mem. Amer. Math. Soc., vol. 71, Amer. Math. Soc., Providence, RI, 1988.
3. H. G. Bothe, *Universal mengen bzuglich der lage im E^n*, Fund. Math. **56** (1964), 203–212.
4. Beverly Brechner, *On the dimension of certain spaces of homeomorphisms*, Trans. Amer. Math. Soc. **121** (1966), 516–548.
5. J. W. Cannon, *A positional characterization of the $(n-1)$-dimensional Sierpiński curve in $S^n (n \neq 4)$*, Fund. Math. **79** (1973), 107–112.
6. K. Kuratowski, *Topology*. II, Academic Press, New York, 1968.
7. S. Lefschetz, *On compact spaces*, Ann. of Math. **2** (1931), 521–538.
8. John C. Mayer, J. Nikiel, and L. G. Oversteegen, *Universal spaces for* **R***-trees*, Trans. Amer. Math. Soc. **334** (1992), 411–432.
9. John C. Mayer and L. G. Oversteegen, *Continuum theory*, Recent Progress in General Topology (M. Hušek and J. van Mill, eds.), North-Holland, Amsterdam, 1992, pp. 453–492.
10. ——, *A topological characterization of* **R***-trees*, Trans. Amer. Math. Soc. **320** (1990), 320–395.
11. K. Menger, *Kurventheorie*, Teubner, Leipzig, 1932.
12. J. Nagata, *Modern dimension theory*, Wiley (Interscience), Groningen, 1965.
13. Togo Nishiura and E. D. Tymchatyn, *Hereditarily locally connected spaces*, Houston J. Math. **2** (1976), 581–599.
14. B. A. Pasynkov, *Partial topological products*, Trudy Moskov. Mat. Obshch. **13** (1965), 136–245; English transl., Trans. Moscow Math. Soc. (1965), 153–271.
15. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, RI, 1942.

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