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COMPLEX EQUILIBRIUM MEASURE AND BERNSTEIN TYPE THEOREMS FOR COMPACT SETS IN $\mathbf{R}^n$

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ABSTRACT. The aim of this paper is to refine and develop further some results from a paper of Bedford and Taylor [Trans. Amer. Math. Soc. **294** (1986), 705–717]. The main result, a Bernstein type theorem, is an improvement of the classical Bernstein inequality

$$|p'(x)| \le (\deg p)(1 - x^2)^{-1/2}(\|p\|^2_{[-1,1]} - p^2(x))^{1/2},$$

from the interval $[-1, 1]$ to the multivariate case.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let E be a compact set in $\mathbf{C}^n$. By u_E we denote the plurisubharmonic extremal function associated with E, defined as follows:

$$u_E(z) = \sup\{u(z) : u \in \mathscr{L}, u \le 0 \text{ on } E\}$$

for $z \in \mathbf{C}^n$, where $\mathscr{L}$ is the Lelong class of plurisubharmonic (briefly, psh) functions in $\mathbf{C}^n$ with logarithmic growth: $u(z) \le \text{const} + \log(1 + |z|)$; see [7].

Denote by u_E^* the upper regularization of the function u_E:

$$u_E^*(z) = \limsup_{w \to z} u_E(w).$$

A subset E of $\mathbf{C}^n$ is said to be pluripolar if there is a psh function u on $\mathbf{C}^n$ such that $E \subset \{u = -\infty\}$.

It is well known (see [12]) that if E is not a pluripolar compact subset of $\mathbf{C}^n$, then $u_E^* \in \mathscr{L}$. In this case we can consider the value of the complex Monge-Ampère operator on the function u_E^*, $(dd^c u_E^*)^n$. Here, for a fixed open set Ω in $\mathbf{C}^n$, $(dd^c \cdot)^n$ is an operator from the space $\mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ of all locally bounded psh function on Ω to the space $B_n(\Omega)$ of positive Borel measures on Ω. In case $u \in \mathrm{PSH}(\Omega) \cap \mathscr{C}^2(\Omega)$ this operator is simply defined by

$$(dd^c u)^n = n!4^n \det\left[\frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}(z)\right] dV(z),$$

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where $dV(z)$ is the usual volume element in $\mathbf{C}^n$. Due to [4] the above operator can be extended to the space $\mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$.

In the special case of u_E^* we have the following.

1.1. **Proposition** ([4]). *If E is a nonpluripolar compact set in $\mathbf{C}^n$, then*

$$(dd^c u_E^*)^n(\mathbf{C}^n \setminus \widehat{E}) = 0$$

where $\widehat{E}$ is the polynomial convex hull of E.

1.2. **Proposition** ([13]). *If E is a nonpluripolar compact subset of $\mathbf{C}^n$, then $(dd^c u_E^*)^n(\widehat{E}) = (2\pi)^n$.*

By the last two propositions $(dd^c u_E^*)^n$ is a Borel measure supported on $\widehat{E}$ whose total mass is equal to $(2\pi)^n$. This measure is called the complex equilibrium measure and is denoted by λ_E (see [5]).

In this paper we will consider the case when E is a compact subset of $\mathbf{R}^n$. Here we treat $\mathbf{R}^n$ as a subset of $\mathbf{C}^n$ such that $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$. The main goal of this paper is to study the density of the complex equilibrium measure and to observe connections between this measure and some classical inequalities for polynomials. Our main result is

1.3. **Theorem.** *Let E be a compact subset of $\mathbf{R}^n$ with nonempty interior. Then $\lambda_E|_{\mathrm{int}(E)} = n!\lambda(x)\,dx$, where $\lambda(x) \ge \mathrm{vol}(\mathrm{conv}\{\frac{1}{\deg p}(1-p^2(x))^{-1/2}\operatorname{grad} p(x) : p \in \mathbf{R}[w],\ \deg p \ge 1,\ |p| < 1$ on $\mathrm{int}(E)\})$ for almost every $x \in \mathrm{int}(E)$.*

The following Bedford-Taylor lemma is very helpful in our considerations.

1.4. **Lemma** ([5]). *Let $u_1, u_2 \in \mathrm{PSH}(\Omega)$ be two given locally bounded functions, such that the Borel measures $(dd^c u_1)^n$, $(dd^c u_2)^n$ are supported on $\Omega \cap \mathbf{R}^n$. If $\{u_1 = 0\} = \{u_2 = 0\} = \Omega \cap \mathbf{R}^n$ and $0 \le u_1 \le u_2$, then*

$$(dd^c u_1)^n \le (dd^c u_2)^n.$$

This result permits us to investigate the equilibrium measure of compact sets in $\mathbf{R}^n$. As an immediate application we have

1.5. **Corollary.** *Let $E \subset F \subset \mathbf{R}^n$ be compact sets. If $\mathrm{int}(E) \ne \varnothing$, then*

$$\lambda_F|_{\mathrm{int}(E)} \le \lambda_E|_{\mathrm{int}(E)}.$$

In this paper we generalize and refine some ideas from the Bedford-Taylor basic paper [5]. In [5] some notions and connections between them are only sketched. Due to [5] we have observed that in investigating the equilibrium measure of compact subsets of $\mathbf{R}^n$ a crucial role is played by the following two notions: the dual set E^* for a subset E of $\mathbf{R}^n$ and the inverse h of the Joukowski function $g(z) = \frac{1}{2}(z + \frac{1}{z})$.

Let us recall some properties of the dual sets and the functions g and h.

Let E be a subset of $\mathbf{R}^n$. Following [11] the dual set E^* for E is defined by

$$E^* = \{y \in \mathbf{R}^n : y \cdot x \le 1 \text{ for each } x \in E\},$$

where "$\cdot$" denotes the usual scalar product in $\mathbf{R}^n$.

The main properties of dual sets are contained in the following.

1.6. **Proposition** (see [11]). *If E is a subset of $\mathbf{R}^n$, then*:

(1) *E^* is a convex, closed subset of $\mathbf{R}^n$ that contains the origin,*
(2) *If E is a compact convex set and $0 \in \operatorname{int}(E)$, then E^* is also compact convex and $0 \in \operatorname{int}(E^*)$. Moreover, $E^{**} = E$.*
(3) *If E is a compact convex set that contains the origin, then*

$$E = \{y \in \mathbf{R}^n : y \cdot x \leq 1 \forall x \in \operatorname{extr}(E^*)(\forall x \in \overline{\operatorname{extr}(E^*)})\}.$$

(4) *If f is a linear automorphism of $\mathbf{R}^n$ and f^* its conjugate (with respect to the usual scalar product), then $(f^{-1}(E))^* = f^*(E^*)$.*

If E is a compact convex subset of $\mathbf{R}^n$, then the function $\chi(x) := \max\{x \cdot y : y \in E\}$, for $x \in \mathbf{R}^n$, is called the support function of the set E. We can write $E = \{x \in \mathbf{R}^n : x \cdot y \leq \chi(y) \quad \forall y \in \mathbf{R}^n\}$. The support function χ is subadditive and positively homogeneous. It is also a seminorm, if a set E is symmetric. Moreover, if $0 \in \operatorname{int}(E)$, then χ is a norm.

If χ is the support function of a compact convex set E, then the set E^* has the representation:

$$E^* = \{x \in \mathbf{R}^n : \chi(x) \leq 1\}.$$

We have the following canonical isomorphism: $l : \mathbf{R}^n \ni x \to \{y \to x \cdot y\} \in (\mathbf{R}^n)^* = \mathscr{L}(\mathbf{R}^n; \mathbf{R})$. If f is a norm in $\mathbf{R}^n$, then the formula $f^*(l(x)) = \max\{x \cdot y / f(y) : y \in S^{n-1}\}$ defines a norm in $(\mathbf{R}^n)^*$. Due to the above isomorphism, the same formula defines a norm in $\mathbf{R}^n$, which is called the dual norm. We denote it by f^*. It is easily seen that for any norm we have $f^{**} = f$. Moreover,

$$f(x) = \max\{x \cdot y / f^*(y) : y \in S^{n-1}\} \quad \text{for every } x \in \mathbf{R}^n, \tag{1.7}$$

where S^{n-1} is the unit Euclidean sphere in $\mathbf{R}^n$. The function $f(\lambda) = |\lambda|$, $\lambda \in \mathbf{R}$, is not differentiable at zero. The same holds for each norm in $\mathbf{R}^n$. In points other than 0, a norm may also not be differentiable. But, it is possible to approximate uniformly any norm by smooth norms on $\mathbf{R}^n \backslash \{0\}$.

1.8. **Proposition.** *If f is a norm in $\mathbf{R}^n$, then there exists a sequence (f_k) of norms in $\mathbf{R}^n$ such that $f_k \in \mathscr{C}^\infty(\mathbf{R}^n \backslash \{0\})$ and $f_k \nearrow f$.*

Proof. By (1.7) we can write $f(x) = \max\{x \cdot y / f^*(y) : y \in S^{n-1}\}$. If σ denotes the usual measure on S^{n-1}, then the following formula holds:

$$\begin{aligned} &\max\{x \cdot y / f^*(y) : y \in S^{n-1}\} \\ &= \lim_{k \to \infty} \left[\int_{S^{n-1}} (x \cdot y / f^*(y))^{2k} \, d\sigma(y) \right]^{1/2k} \\ &= \lim_{k \to \infty} |\varphi(x, \cdot)|_{2k}, \quad \text{where } \varphi : \mathbf{R}^n \times S^{n-1} \ni (x, y) \to x \cdot y / f^*(y) \in \mathbf{R}. \end{aligned}$$

Put $f_k(x) = |\varphi(x, \cdot)|_{2k}$ for $x \in \mathbf{R}^n$. It is clear that f_k is of class $\mathscr{C}^\infty$ on $\mathbf{R}^n \backslash \{0\}$. Since φ is a linear function with respect to x, each function f_k is a norm. Because of Hölder's inequality, the sequence f_k is increasing.

Note also the following connection between the dual norm and the dual set.

1.9. **Proposition.** *If f is a norm in $\mathbf{R}^n$, $r > 0$, then*

$$\{y \in \mathbf{R}^n : f(y) \leq r\}^* = \{y \in \mathbf{R}^n : f^*(y) \leq \tfrac{1}{r}\}.$$

If f is a convex function on a convex open subset U of $\mathbf{R}^n$, then the function $u(x+iy) = f(y)$ is a psh function on the domain $\Omega = \mathbf{R}^n + iU$. Hence every convex function may be regarded as a psh function which is independent of x. In the special case of a norm, by Proposition 1.8, *we have the following important property* (*see* [5]).

1.10. **Proposition.** *If f is a norm in $\mathbf{R}^n$, then $(dd^c f)^n = n!c(f)\,dx$, where*

$$c(f) = \mathrm{vol}(\{f(y) \le 1\}^*) = \mathrm{vol}(\{f^*(y) \le 1\}).$$

It is easy to check the following.

1.11. **Proposition.** *Let a function $l\colon \Omega \times \mathbf{R}^n \to \mathbf{R}_+$ be continuous, where Ω is an open subset of $\mathbf{R}^n$. If for each $x \in \Omega$ the function $l(x,\cdot) = l_x$ is a norm in $\mathbf{R}^n$, then the function $\varphi\colon \Omega \ni x \to \mathrm{vol}(\{l_x^*(y) \le 1\})$ is continuous.*

Now we can formulate the lemma, which is very helpful in computing the complex equilibrium measure.

1.12. **Comparison Lemma.** *Let Ω and D be open sets in $\mathbf{R}^n$ and $\mathbf{C}^n$, respectively, and $\Omega = D \cap \mathbf{R}^n$. Fix a function $u \in \mathrm{PSH}(D) \cap L^\infty_{\mathrm{loc}}(D)$ such that $u^{-1}(0) = \Omega$ and $(dd^c u)^n(D\backslash\Omega) = 0$.*

(a) *Let $l\colon \Omega \times \mathbf{R}^n \to \mathbf{R}_+$ be the function of Proposition* 1.11. *Assume that the following inequality holds uniformly on $K \times (\rho S^{n-1})$, for every compact set $K \subset \Omega$ and small $\rho > 0$:*

$$\limsup_{\alpha\to 0+} \tfrac{1}{\alpha} u(x+i\alpha y) \le l(x,y).$$

Then we have the inequality $(dd^c u)^n|_\Omega \le n!\varphi(x)\,dx$.

(b) *If the measure $(dd^c u)^n|_\Omega$ is absolutely continuous with respect to Lebesgue measure and if we replace the assumption of* (a) *by*

$$\liminf_{\alpha\to 0+} \tfrac{1}{\alpha} u(x+i\alpha y) \ge l(x,y),$$

then we obtain the inequality $(dd^c u)^n|_\Omega \ge n!\varphi(x)\,dx$.

We will apply this lemma to the function u_E for compact sets $E \subset \mathbf{R}^n$ with its appropriate representation. Here the crucial role is played by the function h which is the inverse function to the Joukowski function, $g(z) = \frac{1}{2}(z + \frac{1}{z})$ for $z <\in \mathbf{C}\backslash\{0\}$. This function is univalent on $|z| > 1$ and its inverse is of the form $h(z) = z + (z^2-1)^{1/2}$, if we choose an appropriate branch of the square root. We have the very useful formula (see [3])

$$|h(z)| = h(\tfrac{1}{2}|z+1| + \tfrac{1}{2}|z-1|),$$

for each $z \in \mathbf{C}$, where on the right-hand side we have $h(t) = t + (t^2-1)^{1/2}$ for $t \ge 1$ with positive square root. An easy computation shows the following.

1.13. **Proposition.** (i) *If $\alpha \in (-1,1)$, $\varepsilon > 0$, and $\beta \in \mathbf{R}$, then*

$$\tfrac{1}{\varepsilon}\log|h(\alpha + i\varepsilon\beta)| \le |\beta|(1-\alpha^2)^{-1/2}.$$

(ii) *If $\alpha \in (-1,1)$, $0 < \varepsilon \le \frac{1}{2}$, $\beta \in \mathbf{R}$, and $|\beta| \le 1 - |\alpha|$, then*

$$(1-\varepsilon)|\beta|(1-\alpha^2)^{-1/2} \le \tfrac{1}{\varepsilon}\log|h(\alpha + i\varepsilon\beta)|.$$

In [3], the following property of the extremal function u_E was proved.

1.14. **Proposition.** *For a compact set* $E \subset \mathbf{R}^n$, *we have the equality*

$$u_E(z) = \sup\{\log|h(p(z))|^{1/\deg p} : p \in \mathbf{R}[w],\ \deg p \geq 1,\ \|p\|_E \leq 1\}.$$

We also have

1.15. **Proposition** [8, 1]. *If* E *is a compact, convex, and symmetric subset of* $\mathbf{R}^n$ *with* $0 \in \operatorname{int}(E)$, *then*

$$u_E(z) = \sup\{\log|h(z \cdot w)| : w \in \operatorname{extr}(E^*)\}, \qquad z \in \mathbf{C}^n.$$

The Comparison Lemma 1.12 will be proved in §3. Applications of this result will be given in §4. The proof of Theorem 1.3 is contained in the last part of this paper.

2. Bernstein type theorems

If we put $E = [-1, 1]$, then we obtain from 1.3 that

$$\sup\left\{\frac{1}{\deg p}(1 - p^2(x))^{-1/2}|p'(x)| : p \in \mathbf{R}[w],\ \deg p \geq 1,\ |p| < 1 \text{ on } \operatorname{int}(E)\right\}$$
$$\leq (1 - x^2)^{-1/2},$$

for almost every $x \in (-1, 1)$, which implies Bernstein's inequality for real polynomials:

$$|p'(x)| \leq (\deg p)(1 - x^2)^{-1/2}(\|p\|_E^2 - p^2(x))^{1/2} \quad \text{for every } x \in (-1, 1)$$

(see [6]). This inequality is not as well known as the classical Bernstein-Markov inequality

$$|p'(x)| \leq (\deg p)(1 - x^2)^{-1/2}\|p\|_E.$$

In this context, Theorem 1.3 can be viewed as an extension of the Bernstein result.

In [2] some corollaries to 1.3 and another multivariate version of Bernstein's inequality were given.

Let E be a compact set in $\mathbf{R}^n$ with nonempty interior. Then for each $x \in \operatorname{int}(E)$ we have the following inequalities for a real polynomial p:

$$|D_j p(x)| \leq (\deg p) D_j^+ u_E(x)(\|p\|_E^2 - p^2(x))^{1/2},$$

where $D_j^+ u_E(x) = \liminf_{\varepsilon \to 0+} \frac{1}{\varepsilon} u_E(x + i\varepsilon e_j)$ for $j = 1, \ldots, n$ and $\{e_1, \ldots, e_n\}$ is the standard orthogonal basis in $\mathbf{R}^n$. This inequality is sharp for convex symmetric sets in the sense that in this case we have

$$\sup\{(\deg p)^{-1}|D_j p(x)|(\|p\|_E^2 - p^2(x))^{-1/2}\} = D_j^+ u_E(x), \text{ for } x \in \operatorname{int}(E),$$

where the supremum is being taken over all real polynomials p of degree ≥ 1 with $|p| < \|p\|_E$ on $\operatorname{int}(E)$.

3. Proof of Lemma 1.12

Since in the proof of both parts of the lemma similar arguments can be used we will prove only the first inequality. We start with the following proposition.

3.1. **Proposition.** *With the assumptions* (a) *of Lemma* 1.12(a), *for a fixed* $x_0 \in \Omega$ *and for a fixed, sufficiently small* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $|x - x_0| \le \delta$ *and* $|y| \le \delta$, *then we have*

$$u(x + iy) - l(x_0, y) \le \varepsilon l(x_0, y).$$

Proof. Since the function l is uniformly continuous on $K \times \rho S^{n-1}$, for every compact set $K \subset \Omega$ there exists $\delta > 0$ such that if $x \in \overline{B}(x_0, \delta)$, then $\sup\{u(x + i\alpha y) - l(x, \alpha y): y \in \rho S^{n-1}\} \le \varepsilon\alpha$, for $\alpha < \alpha_0$, and

$$\sup\{|l(x, y) - l(x_0, y)|: y \in \rho S^{n-1}\} \le \varepsilon.$$

Since the function l_x is homogeneous and the norm l_{x_0} is equivalent to the Euclidean norm, $u(x+iy) - l(x_0, y) \le 2M\varepsilon l(x_0, y)$ for $|x - x_0| \le \delta$, $|y| \le \delta$, and $M = 1/\inf\{l_{x_0}(y): y \in \rho S^{n-1}\}$, which completes the proof.

Let us return to the proof of the lemma. Since $\{(1 + \varepsilon)l(x_0, y) \le 1\}^* = (1 + \varepsilon)\{l(x_0, y) \le 1\}^*$, by Lemma 1.4 and Propositions 3.1 and 1.9, we have the inequality

$$(dd^c u)^n|_{B(x_0, \delta)} \le n!(1 + \varepsilon)^n \varphi(x_0)\, dx$$

for a fixed $x_0 \in \Omega$ and small $\varepsilon > 0$. It follows from the above inequality that the measure $(dd^c u)^n$ is absolutely continuous with respect to the Lebesgue measure on Ω. By Radon-Nikodym theorem there exists a nonnegative Borel function λ such that $(dd^c u)^n|_\Omega = \lambda(x)\, dx$. For every $E \in \mathscr{B}(B(x_0, \delta))$ with $\mathrm{vol}(E) > 0$ we have $\frac{1}{\mathrm{vol}(E)} \int_E \lambda(x)\, dx \le n!(1 + \varepsilon)^n \varphi(x_0)$. Hence we get

$$\lambda(x) \le n!(1 + \varepsilon)^n \varphi(x_0)$$

for almost every $x \in B(x_0, \delta)$. By the Lebesgue theorem on density points and by Lusin's theorem, the function λ is approximately continuous for almost every $x \in \Omega$ (see, e.g., [10, p. 148]). Since $\varepsilon > 0$ may be arbitrarily chosen, in every point x_0 of approximate continuity we have $\lambda(x_0) \le n!\varphi(x_0)$. Thus $\lambda \le n!\varphi$ almost everywhere, which completes the proof.

3.2. **Corollary.** *If both assumptions* (a) *and* (b) *of Lemma* 1.12 *are satisfied for the same function* l, *then* $(dd^c u)^n = n!\varphi\, dx$.

4. Some applications of Lemma 1.12 to the complex equilibrium measure

As a first application, we will give a formula for the complex equilibrium measure of a compact, convex symmetric subset of $\mathbf{R}^n$.

Let E be a compact, convex symmetric subset of $\mathbf{R}^n$ with nonempty interior. Then the set E^* has the same properties and, following 1.6, we can write $E = \{x \in \mathbf{R}^n: x \cdot y \le 1 \text{ for each } y \in K\}$, where $K = \overline{\mathrm{extr}(E^*)}$. If $x \in \mathrm{int}(E)$, then $\max\{x \cdot y: y \in K\} < 1$. Moreover, we have

4.1. **Proposition.** *If* $x \in \mathrm{int}(E)$, $y \in \mathbf{R}^n$ *with* $|y| \le \mathrm{dist}(x, \partial E)$, *and* $0 < \varepsilon \le \frac{1}{2}$, *then*

$$(1 - \varepsilon) \max_{w \in K} |y \cdot w|(1 - (x \cdot w)^2)^{-1/2} \le \frac{1}{\varepsilon} u_E(x + i\varepsilon y) \le \max_{w \in K} |y \cdot w|(1 - (x \cdot w)^2)^{-1/2}.$$

Proof. If $|y| \le \mathrm{dist}(x, \partial E)$, then $|y \cdot w| \le 1 - |x \cdot w|$ for every $w \in K$ and the required inequalities follow from 1.13 and 1.15.

4.2. **Corollary.** *Let* $\Omega = \mathrm{int}(E)$, $D = \Omega + i\mathbf{R}^n$, $u = u_E|_D$, *and* $l(x, y) = \max_{w\in K} |y \cdot w|(1-(x\cdot w)^2)^{-1/2}$. *Then the assumptions of* 3.2 *are fulfilled.*

4.3. *Remark.* For a fixed $x \in \mathbf{R}^n$ and $w \in \mathbf{R}^n$ belonging to the strip $\{|x\cdot w| < 1\}$, let $S_x(w) = (1-(x\cdot w)^2)^{-1/2}w$. For $x \in \mathrm{int}(E)$, the condition $l(x, y) \le 1$ is equivalent to $y \in S_x(K)^*$. By Proposition 1.6(3) we get $\{y\colon l(x, y) \le 1\}^* = \mathrm{conv}(S_x(K))$. Moreover, we have

$$\{y\colon l(x, y) \le 1\}^* = \{y \in \mathbf{R}^n\colon y\cdot w \le l(x, w) \text{ for each } w \in S^{n-1}\}.$$

Now we can formulate the main theorem on the complex equilibrium measure of convex symmetric subsets of $\mathbf{R}^n$. As an application we obtain effective formulas in some important cases.

4.4. **Theorem.** *Let* E *be a compact, convex, and symmetric subset of* $\mathbf{R}^n$ *with nonempty interior. Then* $\lambda_E|_{\mathrm{int}(E)} = n!\lambda(x)\,dx$, *where, with the notation of Corollary* 4.2, *we have*

$$\begin{aligned}\lambda(x) &= \mathrm{vol}(\{y \in \mathbf{R}^n\colon y\cdot w \le l(x, w)\forall w \in S^{n-1}\})\\ &= \mathrm{vol}(\mathrm{conv}(S_x(\mathrm{extr}(E^*)))).\end{aligned}$$

Proof. By Corollary 4.2, the above theorem is a consequence of Proposition 4.1.

4.5. *Remark.* Theorem 4.4 refines a result of Bedford and Taylor [5], since we can replace $S_x(E^*)$ in [5] by $S_x(\mathrm{extr}(E^*))$. Hence, our theorem is better in a case of a convex symmetric polyhedron. It seems to be true that the boundary of a compact, convex subset of $\mathbf{R}^n$ is pluripolar in $\mathbf{C}^n$. This situation holds for convex polyhedrons and in any of the examples below. However, we do not know of any proof in the general case.

4.6. **Example.** Let $I_n = [-1, 1]^n$. Then $\mathrm{extr}(E^*) = \{\pm e_1, \dots, \pm e_n\}$. Hence $\lambda(x) = \mathrm{vol}(\mathrm{conv}\{\pm(1-x_1^2)^{-1/2}e_1, \dots, \pm(1-x_n^2)^{-1/2}e_n\})$ and one can easily calculate that $\lambda_{I_n} = 2^n(1-x_1^2)^{-1/2}\cdot\dots\cdot(1-x_n^2)^{-1/2}\,dx$. This formula was obtained in [5] in a different way.

4.7. **Example.** Let $E = B_n$ be the closed unit Euclidean ball in $\mathbf{R}^n$. Then $u_E(z) = \frac{1}{2}\log h(|z|^2 + |z^2-1|)$ (see [8] and [1]) and it is easy to verify that $l(x, y) = (y^2 + (y\cdot x)^2/(1-x^2))^{1/2}$. Thus

$$F_x = \{y\colon l(x, y) \le 1\} = \{y^2 + (y\cdot x)^2(1-x^2)^{-1} \le 1\}.$$

If $x = 0$, then $F_0 = B_n = F_0^*$ and $\lambda(0) = \mathrm{vol}(B_n)$. Assume that $x \ne 0$, and complete the vector $x/|x|$ by $u_1, \dots, u_{n-1}$ to an orthonormal basis in $\mathbf{R}^n$. Let f be the orthonormal automorphism which is given by formula $f(\mu) = \mu_1u_1 + \dots + \mu_{n-1}u_{n-1} + \mu_n x/|x|$. We have

$$\begin{aligned}f^{-1}(F_x) &= \{\mu_1^2 + \dots + \mu_n^2 + \mu_n^2x^2/(1-x^2) \le 1\}\\ &= \{\mu_1^2 + \dots + \mu_{n-1}^2 + \mu_n^2/(1-x^2) \le 1\},\end{aligned}$$

which implies that $F_x = f(\{\mu_1^2 + \dots + \mu_{n-1}^2 + \mu_n^2/(1-x^2) \le 1\})$. Thus, by Proposition 1.6, we get

$$\begin{aligned}F_x^* &= f^{*-1}(\{\mu_1^2 + \dots + \mu_{n-1}^2 + \mu_n^2/(1-x^2) \le 1\}^*)\\ &= f^{*-1}(\{\mu_1^2 + \dots + \mu_{n\ \ 1}^2 + \mu_n^2(1-x^2) \le 1\}),\end{aligned}$$

and finally

$$\begin{aligned}\operatorname{vol}(F_x^*) &= |\det f^{*-1}| \operatorname{vol}(\{\mu_1^2+\cdots+\mu_{n-1}^2+\mu_n^2(1-x^2)\le 1\})\\ &= |\det f^{-1}|(1-x^2)^{-1/2}\cdot \operatorname{vol}(B_n)\\ &= \operatorname{vol}(B_n)(1-x^2)^{-1/2}.\end{aligned}$$

Consequently we obtain a simple formula

$$\lambda_{B_n} = n!\operatorname{vol}(B_n)(1-x^2)^{-1/2}\,dx,$$

which is the main result from Lundin's paper [9].

4.8. **Example.** Let $E=\{x\in\mathbf{R}^n\colon x_1,\dots,x_n\ge 0,\ x_1+\cdots+x_n\le 1\}=S_n$ is the standard simplex in $\mathbf{R}^n$. We have

$$u_E(z)=\log h(|z_1|+\cdots+|z_n|+|z_1+\cdots+z_n-1|),$$

and an easy calculation shows that

$$\lambda_{S_n}=n!\operatorname{vol}(B_n)(x_1\cdot\cdots\cdot x_n)^{-1/2}(1-x_1-\cdots-x_n)^{-1/2}\,dx.$$

4.9. **Example.** Let $E=B_2\times([-b,-a]\cup[a,b])\subset\mathbf{R}^3$, where $0<a<b$. Using a similar method to that of Lemma 1.12, we obtain

$$\lambda_E=4\pi(1-x^2-y^2)^{-1/2}|z|(b^2-z^2)^{-1/2}(z^2-a^2)^{-1/2}\,dx\,dy\,dz.$$

4.10. **Example.** Let $E=\{(x,y)\in\mathbf{R}^2\colon x,y\ge 0,\ x^{1/2}+y^{1/2}\le 1\}$. By applying a result of Klimek (see, e.g., [7, Theorem 5.3.1]) to the set $[-1,1]\times[-1,1]$ and the mapping $f(z_1,z_2)=\frac14((z_1-z_2)^2,\ (z_1+z_2)^2)$, we find that

$$u_E(z)=2\max(\log|h(\sqrt{z_1}+\sqrt{z_2})|,\log|h(\sqrt{z_1}-\sqrt{z_2})|).$$

Then one can calculate that

$$\lambda_E=8\frac{1}{\sqrt{x_1x_2}}\frac{1}{(1-(\sqrt{x_1}+\sqrt{x_2})^2)^{1/2}}\frac{1}{(1-(\sqrt{x_1}-\sqrt{x_2})^2)^{1/2}}\,dx.$$

We omit here the details of the proofs of Examples 4.8–4.10.

5. Proof of theorem 1.3

We start with the following remarks. If $p\in\mathbf{C}[z]$, $\deg p=k\ge 1$, then Taylor's formula yields for every $a\in\mathbf{C}^n$:

$$p(z)=\sum_{|\alpha|\le k}\frac{1}{\alpha!}D^\alpha p(a)(z-a)^\alpha.$$

If $p\in\mathbf{R}[z]$, then for $z=x+i\varepsilon y$, $a=x$ we get

$$\begin{aligned}p(x+i\varepsilon y)&=p(x)+i\varepsilon y\cdot\operatorname{grad}p(x)+\sum_{2\le|\alpha|\le k}\frac{1}{\alpha!}D^\alpha p(x)(i\varepsilon)^{|\alpha|}y^\alpha\\ &=p(x)+P_\varepsilon(x,y)+i\varepsilon y\cdot\operatorname{grad}p(x)+i\varepsilon Q_\varepsilon(x,y),\end{aligned}$$

where P_ε, $Q_\varepsilon\in\mathbf{R}[z]$. Assume that:

$$\sum_{1\le|\alpha|\le k}\frac{1}{\alpha!}|D^\alpha p(x)|\,|y|^{|\alpha|}\le 1-|p(x)|.$$

Then it follows that

$$|y\cdot\operatorname{grad}p(x)+Q_\varepsilon(x,y)|\le 1-|p(x)+P_\varepsilon(x,y)|$$

and one can easily verify the following.

5.1. **Lemma.** *Let* $u_k(z) = \max_{1\le m\le k}(1/\deg p_m)\log|h(p_m(z))|$, *where* p_k *is a sequence of all real polynomials with rational coefficients, such that* $|p| < 1$ *on* $\mathrm{int}(E)$. *Then for every* $x \in \mathrm{int}(E)$ *and* $y \in \mathbf{R}^n$ *the limit*

$$\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} u_k(x + i\varepsilon y) = l_k(x, y)$$

$$= \max_{1\le m\le k} \frac{1}{\deg p_m}|y \cdot \operatorname{grad} p_m(x)|(1 - p_m^2(x))^{-1/2}$$

exists, the convergence being uniform on any set $K \times S^{n-1}$, *where* K *is an arbitrary compact subset of* $\mathrm{int}(E)$. *Moreover, for all* $x \in \mathrm{int}(E)$, *the function* $l_k(x, \cdot)$ *is a norm, provided* k *is sufficiently large.*

We shall need the following

5.2. **Lemma.** *If* E *is a compact subset of* $\mathbf{R}^n$ *with* $\mathrm{int}(E) \neq$, *then the measure* $\lambda_E|_{\mathrm{int}(E)}$ *is absolutely continuous with respect to measure* $dx|_{\mathscr{B}(\mathrm{int}(E))}$.

Proof. The lemma is an immediate consequence of Corollary 1.5 and Example 4.7.

If k is fixed and sufficiently large, then by Comparison Lemma 1.12 we obtain

$$n \operatorname{vol}\left(\operatorname{conv}\left\{\frac{1}{\deg p_j}(1 - p_j^2(x))^{-1/2}\operatorname{grad} p_j(x) \colon 1 \le j \le k\right\}\right) dx \le \lambda_E|_{\mathrm{int}(E)}.$$

The Lebesgue monotone convergence theorem yields

$$n! \operatorname{vol}\left(\operatorname{conv}\left\{\frac{1}{\deg p}(1 - p^2(x))^{-1/2}\operatorname{grad} p(x) : p \in \mathbf{Q}[z],\ |p| < 1 \text{ on } \mathrm{int}(E)\right\}\right) dx \le \lambda_E|_{\mathrm{int}(E)}.$$

Now observe that $\operatorname{vol}(\operatorname{conv}(A)) = \operatorname{vol}(\operatorname{conv}(\overline{A})) = \operatorname{vol}(\overline{\operatorname{conv}(A)})$ for any bounded subset A of $\mathbf{R}^n$, which completes the proof.

5.3. *Remark.* In the case of a convex, compact symmetric subset of $\mathbf{R}^n$ we can replace the inequality $\geq$ in Theorem 1.3 by the equality. We conclude this fact from Theorem 4.4. We conjecture that the equality holds in the general case too.

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