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THE DISCRETE NATURE OF THE PALEY-WIENER SPACES

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ABSTRACT. The Shannon Sampling Theorem suggests that a function with bandwidth π is in some way determined by its samples at the integers. In this work we make this idea precise for the functions in the Paley-Wiener space E^p. For $p > 1$, we make a modest contribution, but the basic result is implicit in the classical work of Plancherel and Pólya (1937). For $0 < p \leq 1$, we combine old and new results to arrive at a characterization of E^p via the discrete Hilbert transform. This indicates that for such entire functions to belong to $L_p(\mathbf{R}, dx)$, not only is a certain rate of decay required, but also a certain subtle oscillation.

1. INTRODUCTION

In this paper we study, for $0 < p$, the space E^p_τ of entire functions f of finite exponential type τ for which

$$||f||_p^p = \int_{-\infty}^{+\infty} |f(x)|^p dx < +\infty.$$

E^p_τ is clearly a subspace of $L_p(\mathbf{R}, dx)$, so $|| \ ||_p$ is a norm for $1 \leq p$ and a quasinorm for $0 < p < 1$. Recall that an entire function f is of exponential type τ if $f(z) = \mathcal{O}(e^{((\tau+\varepsilon)|z|)})$ for all $\varepsilon > 0$.

For the sequel, we essentially consider $\tau = \pi$, as the other cases are handled by a change of variables. Henceforth, $E^p_\pi = E^p$. Our definition of E^p is motivated by a classical theorem of Paley and Wiener.

Theorem 1 (Paley and Wiener). *For an entire function f to belong to E^2, it is necessary and sufficient that there exist $\psi \in L_2([-\pi, \pi])$ such that*

$$f(z) = \int_{-\pi}^{\pi} \psi(t) e^{itz} dt.$$

Basic facts about entire functions can be found in [1]; in particular, for f in E^p, $|f(x)| \to 0$, as $|x| \to +\infty$. This allows for the observation that, unlike the $L_p(\mathbf{R}, dx)$ spaces, the E^p spaces are nested: $E^p \subseteq E^q$, if $0 < p \leq q$.

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Many facts about E^p, $0 < p < 2$, follow from known facts about E^2. (E^2 is denoted PW by some authors, e.g., [7].)

A brief review of some of these facts: E^2 is the isometric image of $L_2([-\pi, \pi])$ under the inverse Fourier transform and is therefore a Hilbert space. Generally speaking, a function whose Fourier transform is supported in an interval is said to be *band-limited*; such functions are interpreted as signals, with no frequencies outside the "band". E^2 seems to play a significant role in signal processing applications [5]. Central to the E^2 theory is the so-called *sinc* function

$$\operatorname{sinc}(z) = \frac{\sin \pi z}{\pi z}.$$

Since $\operatorname{sinc}(z-n)$ is the image of $e^{-int}/\sqrt{2\pi}$ under the inverse transform, the collection $\{\operatorname{sinc}(z-n)\}_{n\in\mathbf{Z}}$ is an orthonormal basis of E^2.

The *cardinal series* of a function f is

$$f(x) = \sum_{n=-\infty}^{+\infty} f(n) \operatorname{sinc}(x-n).$$

Many facts about the history of the cardinal series and especially its place in communication theory can be found in the comprehensive article of J. R. Higgins [5]. As we will see, for $p > 1$ the sinc functions play the same role as the standard unit vectors in l_p. Although the sinc functions do not belong to E^p, for $0 < p \leq 1$, they are still central to our results. A bit of notation: In this paper, l_p will denote the space of p-summable sequences indexed on the integers. Also, the term *samples* of a function f will always refer to the sequence $\{f(n)\}_{n\in\mathbf{Z}}$.

2. E^p IS A QUASI-BANACH SPACE

Although E^p is clearly a subspace of $L_p(\mathbf{R}, dx)$, it does not seem to have been noticed that E^p is complete, for values of p other than 2. To show that E^p is closed, it suffices to prove that convergence in E^p forces uniform convergence on compact subsets of $\mathbf{C}$ and preserves type. This and more will follow from the following results of Plancherel and Pólya [6].

Theorem 2 (Plancherel and Pólya). *Let $p, \tau > 0$ and $f \in E^p_\tau$.*

(i) *For $y \in \mathbf{R}$*

$$\int_{-\infty}^{+\infty} |f(x+iy)|^p \, dx \leq e^{p\tau|y|} \int_{-\infty}^{+\infty} |f(x)|^p \, dx.$$

(ii) *There exists a constant $A > 0$, which depends only on τ and p so that*

$$\sum_{n=-\infty}^{+\infty} |f(n)|^p \leq A \int_{-\infty}^{+\infty} |f(x)|^p \, dx.$$

Let $z_0 = x_0 + iy_0 \in \mathbf{C}$ and denote $f_{z_0}(u) = f(u+z_0)$. If $f \in E^p$, then f_{z_0}

also belongs to E^p. Applying (i) and (ii) we see that

$$\begin{aligned}
|f(z_0)|^p &= |f_{z_0}(0)|^p \\
&\le \sum_{-\infty}^{\infty} |f_{z_0}(k)|^p \\
&\le B \int_{-\infty}^{\infty} |f_{z_0}(t)|^p \, dt \\
&= B \int_{-\infty}^{\infty} |f_{iy_0}(t + x_0)|^p \, dt \\
&= B \int_{-\infty}^{\infty} |f_{iy_0}(t)|^p \, dt \\
&= B \int_{-\infty}^{\infty} |f(t + iy_0)|^p \, dt \\
&\le B e^{p\pi|y|} \int_{-\infty}^{\infty} |f(t)|^p \, dt.
\end{aligned}$$

Consequently, for $f \in E^p$,

$$|f(z_0)| \le B e^{\pi|y|} \|f\|_p.$$

From this, we see that if (f_n) is a Cauchy sequence in E^p, then it is Cauchy with respect to the topology of uniform convergence on compacta. Consequently, the limit function is not only in $L_p(\mathbf{R})$, but is entire and of exponential type π. These observations yield the following result.

Theorem 3. *For $0 < p$, E^p is complete with respect to the $\| \ \|_p$ quasinorm.*

At this point, perhaps it is worthwhile to list some of the bounded operators on E^p which are of natural interest. It is obvious that real translation maps E^p isometrically into itself, and it follows from Theorem 2 that complex translation also maps E^p boundedly into itself. The map $f \to f_c$, where $f_c(z) = f(cz)$, is a bounded map into E^p for $|c| \le 1$, but in general f_c may not belong to E^p, for $|c| > 1$. Also, it follows from the work of Plancherel and Pólya in [6] that differentiation is a bounded operator from E^p into itself. Finally, it should be observed that part (i) of Theorem 2 implies that the map $f \to e^{i\pi z} f$ is an isometry from E^p into H^p of the upper half-plane.

3. E^p IS ISOMORPHIC TO l_p, $p > 1$

At the heart of our subsequent results is the following classical result of Plancherel and Pólya [6].

Theorem 4 (Plancherel and Pólya). *Let $p, \tau > 0$ and let $f \in E^p_\tau$.*

(i) *If $\tau < \pi$, then there exists a constant $B > 0$ which depends only on τ and p so that*

$$\int_{-\infty}^{+\infty} |f(x)|^p dx \le B \sum_{n=-\infty}^{+\infty} |f(m)|^p.$$

(ii) *If* $\lim_{z\to\infty} f(z) e^{-\pi|z|} = 0$ *and if* $1 < p$, *then* (i) *holds and the constant B depends only on p.*

Now Plancherel and Pólya proved that if a function f of exponential type τ also belongs to $L_p(\mathbf{R})$, then f satisfies $\lim_{z\to\infty} f(z)e^{-\tau|z|} = 0$. Theorems 2 and 4 allowed them to make the following observation.

Corollary 1. *Let* $1 < p$. *There exist constants* C_1 *and* C_2 (*depending only on* p) *such that for all functions in* E^p

$$C_1 \sum_{n=-\infty}^{+\infty} |f(n)|^p \le \int_{-\infty}^{+\infty} |f(x)|^p dx \le C_2 \sum_{n=-\infty}^{+\infty} |f(n)|^p.$$

Plancherel and Pólya used this fact to prove that for a function f in E^p, $p > 1$, the cardinal series

$$\sum_{n=-\infty}^{\infty} f(n)\,\mathrm{sinc}(z-n)$$

converges *en moyenne d'ordre* p *vers* f [6]. Conversely, it is clear from the corollary and Theorem 3 that given a sequence $\{\alpha_n\}$ in l_p one can show that the resulting cardinal series

$$F(z) = \sum_{n=-\infty}^{\infty} a_n\,\mathrm{sinc}(z-n)$$

represents a unique function in E^p, with samples $\{\alpha_n\}$. It is also evident that the sinc functions form a basis, in fact, an unconditional basis for E^p, $p > 1$. These remarks, together with Corollary 1, yield the following result.

Theorem 5. *Let* $p > 1$. E^p *is isomorphic to* l_p *via the mapping* $f \to \{f(n)\}_{n\in\mathbf{Z}}$.

This result has an obvious consequence.

Corollary 2. *For* $p > 1$, $(E^p)^* \cong E^q$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

We see that the action of a linear functional $\phi \in (E^p)^*$, $\phi \sim g \in E^q$, is given by

$$\phi(f) = \sum_{n\in\mathbf{Z}} f(n)\overline{g(n)} = \int_{-\infty}^{+\infty} f(x)\overline{g(x)}dx$$

for all $f \in E^p$.

4. E^p IS ISOMORPHIC TO THE DISCRETE HARDY SPACE $H^p(\mathbf{Z})$, $0 < p \le 1$

In many situations, $p = 1$ is a critical value. Recall from the Hardy space theory that since the Hilbert transform is bounded on L_p, $p > 1$, H^p and L_p are essentially the same. However, for $0 < p \le 1$, H^p turns out to be a proper closed subspace of L_p consisting of functions which not only satisfy the size condition but also possess a certain type of cancellation. Indeed, H^p consists of those functions in L_p for which the Hilbert transform (viz., the conjugation operator) is bounded (e.g., see [4] or [2]).

We have found that a parallel situation exists for E^p with respect to l_p. Now $l_p = L_p(\mathbf{Z}, d\sigma)$ for $\sigma =$ counting measure; so, by analogy with the standard theory (and at the risk of overusing the Hardy space notation and terminology), we define the *discrete Hardy space*, $H^p(\mathbf{Z})$, $0 < p < \infty$, to consist of those sequences $\alpha = \{\alpha_k\} \in l_p$ which satisfy

$$\sum_{k \in \mathbf{Z}} \left| \sum_{n \neq k} \frac{\alpha_n}{k-n} \right|^p < +\infty.$$

Thus $H^p(\mathbf{Z})$ is the subspace of l_p consisting of those sequences $\alpha = \{\alpha_n\}$ for which the *discrete Hilbert transform* also belongs to l_p.

The discrete Hilbert transform, H, of a sequence $\alpha = \{\alpha_n\}$ is defined by

$$H(\alpha)(k) = \sum_{n \neq k} \frac{\alpha_n}{k-n}.$$

Note that for any noninteger $c \in \mathbf{R}$, $H_c(\alpha)(k) = \sum_{n \in \mathbf{Z}} \alpha_n/(k-n+c)$ yields the same class of sequences. $H_c(\alpha)$ is convolution of the sequence α with the kernel $1/(n+c)$. For the sequel, we will use H_c with $c = \frac{1}{2}$, which we will (by a small abuse of notation) denote by H.

$H^1(\mathbf{Z})$ is mentioned by Coifman and Weiss [2, p. 622] as an example of a Hardy space, $H^p(\mathbf{X})$, associated with a space $\mathbf{X}$ of homogeneous type; these spaces are the result of extending the atomic decomposition theory for the classical Hardy spaces to more general settings. A caveat regarding notation: $H^p(\mathbf{X})$ is defined atomically in [2]; thus it is not obvious that $H^p(\mathbf{Z})$ as defined above coincides with the corresponding atomic space of the same label in [2], although it is easily seen to contain the atomic space. (Coifman and Weiss suggest that the two are the same [2]; we shall consider the connection in a later paper.)

A priori, it would appear that, for $p > 1$, $H^p(\mathbf{Z})$ and l_p are different. However, Plancherel and Pólya [6] proved that if a sequence $\alpha = \{\alpha_n\} \in l_p$ for $p > 1$, then there is a constant $C > 0$, so that

$$||H(\alpha)||_{l_p} \leq C||\alpha||_{l_p}.$$

(A discrete version of the M. Riesz theorem.) Thus, for $p > 1$, $H^p(\mathbf{Z})$ and l_p coincide.

For $0 < p \leq 1$, we define an obvious quasinorm on $H^p(\mathbf{Z})$. For $\alpha = \{\alpha_n\}_{n \in \mathbf{Z}}$,

$$||\alpha||_{H^p} = ||\alpha||_{l_p} + ||H(\alpha)||_{l_p}.$$

That $H^p(\mathbf{Z})$ is complete with respect to this quasinorm will follow from subsequent results.

We recall Plancherel and Pólya's inequality for functions of type strictly less than π.

$$\int_{-\infty}^{+\infty} |f(x)|^p dx \leq B \sum_{n=-\infty}^{+\infty} |f(m)|^p.$$

As we recall, for $p > 1$ this inequality holds for functions of type equal to π, provided it is known that the function lies in E^p. For $0 < p \leq 1$, the inequality cannot hold in general for functions of type equal to π, even for functions belonging to E^p. For example, let

$$g_n(z) = \frac{n \sin(\pi z)}{\pi z(z-n)}.$$

Now

$$g_n = -\frac{\sin \pi x}{\pi x} + \frac{\sin \pi x}{\pi(x-n)},$$

so that for x between 0 and n, $|g_n(x)| \geq |\frac{\sin \pi x}{\pi x}|$. Consequently

$$\int_{-\infty}^{\infty} |g_n(x)|^p dx \geq \int_0^{|n|} \left|\frac{\sin \pi x}{\pi x}\right|^p dx,$$

so that $\|g_n\|_p \sim |n|^{\frac{1}{p}-1}$, for $\frac{1}{2} < p < 1$ and $\sim \log|n|$ for $p = 1$, even though $\|\{g_n(k)\}\|_{l_p} = 2^{\frac{1}{p}}$ for all n. Similar examples can be constructed for $0 < p \leq \frac{1}{2}$. In particular, Plancherel and Pólya's inequality reveals a way to test if a function of type π belongs to E^p; that is, we need only determine whether the samples $\{f(\frac{n}{2})\}$ belong to l_p, due to the fact that $f(\frac{z}{2})$ is of type $\frac{\pi}{2}$. It is this simple observation that allows us to show the connection between E^p and $H^p(\mathbf{Z})$.

Theorem 6. *Let $0 < p \leq 1$. If f belongs to E^p, then $\{(-1)^n f(n)\}$ belongs to $H^p(\mathbf{Z})$. Conversely, if $\{\alpha_n\}$ belongs to $H^p(\mathbf{Z})$, there is a unique $f \in E^p$ such that $f(n) = (-1)^n \alpha_n$.*

Proof. Let $f \in E^p$. f has a cardinal series representation

$$f(z) = \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc}(z-n).$$

(Since f is also in E^2, the cardinal series converges uniformly on compact subsets of $\mathbf{C}$.)

From Theorem 4(i), $\{f(\frac{n}{2})\} \in l_p$. For even n, we simply recover the original samples of f, which thus belong to l_p. For odd n, $n = 2k+1$, $k \in \mathbf{Z}$,

$$\begin{aligned} f\left(\frac{2k+1}{2}\right) &= \sum_m f(m) \operatorname{sinc}\left(\frac{2k+1}{2} - m\right) = \sum_m f(m) \frac{\sin \pi(\frac{2k+1}{2} - m)}{\pi(\frac{2k+1}{2} - m)} \\ &= \frac{\sin(\frac{2k+1}{2})\pi}{\pi} \sum_m \cos(m\pi) f(m) \frac{1}{k - m + \frac{1}{2}} \\ &= \frac{(-1)^k}{\pi} \sum_m \frac{(-1)^m f(m)}{k - m + \frac{1}{2}} = \frac{(-1)^k}{\pi} H(\{(-1)^m f(m)\})(k). \end{aligned}$$

Since $\{f(\frac{2k+1}{2})\} \in l_p$, it follows that the Hilbert transform of $\{(-1^n) f(n)\} \in l_p$, whereby $\{(-1)^n f(n)\} \in H^p(\mathbf{Z})$.

Next suppose $\{\alpha_n\} \in H^p(\mathbf{Z})$. We form the cardinal series

$$g(z) = \sum_{n=-\infty}^{\infty} (-1)^n \alpha_n \operatorname{sinc}(z-n).$$

Since $\{\alpha_n\} \in l_p$, g is at least in E^2, and thus we know that the cardinal series converges uniformly on compacta. Now $g(\frac{z}{2})$ is of type $\frac{\pi}{2}$; thus we may apply Theorem 4(i). The above calculations with the cardinal series show that the sequence of samples of $g(\frac{z}{2})$ at the even integers is $\{\alpha_k\}$, and the sequence of samples of $g(\frac{z}{2})$ at the odd integers is $\{\frac{(-1)^k}{\pi} H(\{\alpha_n\})(k)\}_{k \in \mathbf{Z}}$. Consequently the sequence $\{g(\frac{n}{2})\}$ belong to l_p, whereby $g \in E^p$. □

This proof shows that we can map E^p onto $H^p(\mathbf{Z})$ via the map $f \to \{(-1)^n f(n)\}$. For a function f in E^p,

$$\sum_{n=-\infty}^{\infty} \left|f\left(\frac{n}{2}\right)\right|^p = \sum_{k=-\infty}^{\infty} |f(k)|^p + \sum_{k=-\infty}^{\infty} \left|f\left(\frac{2k+1}{2}\right)\right|^p$$

$$= \sum_{k=-\infty}^{\infty} |f(k)|^p + \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \left| \sum_{m=-\infty}^{\infty} \frac{(-1)^m f(m)}{k-m+\frac{1}{2}} \right|^p .$$

It follows from Theorem 4 and the above proof that there are constants $C_1, C_2 > 0$ so that

$$C_1 ||\{(-1)^n f(n)\}||_{H^p} \leq ||f||_p \leq C_2 ||\{(-1)^n f(n)\}||_{H^p}$$

for all $f \in E^p$.

Thus we see that the $H^p(\mathbf{Z})$ quasinorm is equivalent to the E^p quasinorm, so that the map from E^p onto $H^p(\mathbf{Z})$ is continuous, thereby yielding the following result.

Theorem 7. *For $0 < p \leq 1$, E^p is isomorphic to $H^p(\mathbf{Z})$.*

5. Comments

It is clear that sequences in $H^p(\mathbf{Z})$, $0 < p \leq 1$, must sum to zero. Consequently, for a function in E^p, $\sum_{n\in\mathbf{Z}}(-1)^n f(n) = 0$; this also follows from well-known facts from classical harmonic analysis. For a function f in E^p, the Fourier transform $\hat{f}$ is continuous on $\mathbf{R}$ and 0 off of $[-\pi, \pi]$. The above summation simply reflects the fact that $\hat{f}(\pm\pi) = 0$. In fact, it is the cancellation that distinguishes $H^p(\mathbf{Z})$ from l_p and, consequently, essentially what distinguishes E^p from L_p, for $0 < p \leq 1$. As for $H^p(\mathbf{Z})$, membership in E^p requires progressively greater cancellation (actually, oscillation) for progressively smaller values of p. We will further examine this and other properties of E^p in a subsequent paper [3].

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