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# SUBALGEBRAS OF $C(M(H^\infty))$

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ABSTRACT. In this paper we will provide a natural proof that if $f$ is a bounded function that is harmonic but not analytic on $\mathbb{D}$, then $H^\infty(\mathbb{D})[f]$ contains $C(\overline{\mathbb{D}})$. We will also give a necessary and sufficient condition for a closed subalgebra of $C(M)$ to contain $C(\overline{\mathbb{D}})$.

## 1. INTRODUCTION

Let $H^\infty(\mathbb{D})$ denote the algebra of bounded analytic functions on the open unit disk $\mathbb{D}$. The maximal ideal space of $H^\infty(\mathbb{D})$ will be denoted by $M$. Under the Gelfand transform, we think of $H^\infty(\mathbb{D})$ as a closed subalgebra of $C(M)$. For information about $M$, see [5, Chapter 10] or [4, Chapters V, VIII, and X].

Every bounded harmonic function $f$ on $\mathbb{D}$ can be uniquely extended to be continuous on $M$ [6, Lemma 4.4]. Note that the uniqueness of the extension is due to the fact that $\mathbb{D}$ is dense in $M$. Thus $H^\infty(\mathbb{D})[f]$, the norm closed subalgebra of $L^\infty(\mathbb{D})$, can be considered as a closed subalgebra of $C(M)$; we use the same notation for a function in $H^\infty(\mathbb{D})[f]$ regardless of whether we view the function as being defined on $\mathbb{D}$ or $M$.

In general if $f \in C(\overline{\mathbb{D}})$ then $H^\infty(\mathbb{D})[f]$ need not contain $C(\overline{\mathbb{D}})$. For example, take $f$ to be zero on $|z| < \frac{1}{2}$ and equal to $|z| - \frac{1}{2}$ on $\frac{1}{2} \le |z| \le 1$. The main result in [1] states that, if $f$ is a bounded function that is harmonic but not analytic on $\mathbb{D}$ then $H^\infty(\mathbb{D})[f]$ contains $C(\overline{\mathbb{D}})$.

In this paper we will provide a natural proof of this theorem, and give a necessary and sufficient condition for a closed subalgebra of $C(M)$ to contain $C(\overline{\mathbb{D}})$.

## 2. SUBALGEBRAS OF $C(M)$ THAT CONTAIN $C(\overline{\mathbb{D}})$

If $A$ is a function algebra on $X$, then the essential set $\mathscr{E}$ of $A$ is the smallest closed subset of $X$ such that $A$ contains every continuous function on $X$ which vanishes on $\mathscr{E}$ [3, p. 145]. A useful property of the essential set $\mathscr{E}$ of $A$ is that it is equal to the closure $U_\alpha E_\alpha$, where $E_\alpha$ runs over all nontrivial maximal antisymmetric sets for $A$ [8, p. 65]. For information about antisymmetric sets and the essential set of an algebra, see [3, 8]. The maximal

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ideal space of $A$ is denoted by $M(A)$. Let $P$ denote the natural projection of $M(A)$ onto $M$.

**Theorem 1.** *Let $A$ be a closed subalgebra of $C(M)$ containing $H^\infty(\mathbb{D})$ and let $\mathscr{E}$ be its essential set. Then the following conditions are equivalent*:

(1) $P\mathscr{E} \subset M\backslash\mathbb{D}$;

(2) $C(\overline{\mathbb{D}}) \subset A$.

*Proof*. Assume (1) and let us prove that $\overline{z} \in A$. Let $E$ be a nontrivial MAS (maximal antisymmetric set) in $X \stackrel{\text{def}}{=} M(A)$. By assumption there is a neighborhood $V$ of $PE$ disjoint with $\varphi_0$, where $\varphi_0(f) \stackrel{\text{def}}{=} f(0)$. So $U = P^{-1}V$ is disjoint with $P^{-1}\varphi_0$. As $U$ is a neighborhood of a weak peak set $E$, there exists $f \in A$ such that

$$|f|\,|X\backslash U < 1; \qquad f|E = 1.$$

If $\varphi \in P^{-1}\varphi_0$ then $\varphi \in X\backslash U$ and thus $(1-f)(\varphi) \neq 0$.

This implies [3, Corollary 1.2.13] that there exist functions $g$ and $h$ in $A$ such that $zg + (1-f)h = 1$. In particular, $zg = 1$ on $E$. Since $PE \subset M\backslash\mathbb{D}$ then $|z| = 1$ on $E$. Thus $\overline{z}_{|E} = (\frac{1}{z})_{|E} = g_{|E} \in A_{|E}$. By the Bishop antisymmetric decomposition theorem [3, Theorem 2.7.5] we get $\overline{z} \in A$, and so by the Stone-Weierstrass theorem we conclude that $C(\overline{\mathbb{D}}) \subset A$, completing the proof that (2) implies (1).

Conversely, assume that $C(\overline{\mathbb{D}}) \subset A$. Since $z$ and $\overline{z}$ are in $A$ then it is easy to see that for any nontrivial maximal antisymmetric set $E$ for $A$ we get $z_{|E} = \alpha$, where $\alpha$ is a constant number with $|\alpha| \leq 1$. There are two possibilities:

(1) $PE \subset M\backslash\mathbb{D}$;

(2) $PE \subset \{\alpha\}$, $\alpha \in \overline{\mathbb{D}}$.

The second possibility is excluded by the following lemma which also finishes the proof of Theorem 1.

**Lemma.** *Let $H^\infty(\mathbb{D}) \subset A \subset C(M)$ and $C(\overline{\mathbb{D}}) \subset A$. Then for each $\alpha \in \mathbb{D}$, $P^{-1}\alpha = \alpha$.*

*Proof*. Let $a \in A$, $a(\alpha) = 0$, and $\varphi \in P^{-1}\alpha$. We need to prove that $\varphi(a) = 0$. Let us consider the sequence of functions

$$c_n(z) = \begin{cases} 0, & |z-\alpha| \leq n^{-1}, \\ 1, & |z-\alpha| \geq 2n^{-1}, \end{cases}$$

$0 \leq c_n \leq 1$, $c_n \in C(\overline{\mathbb{D}})$. Then uniformly on $M$, $c_n a$ converges to $a$. Also $c_n a \in A$ by our assumption. Thus it is enough to show that $\varphi(c_n a) \to 0$. This is a simple calculation:

$$\begin{aligned} \varphi(c_n a) &= \varphi\left(\frac{c_n}{z-\alpha}a \cdot (z-\alpha)\right) \\ &= \varphi\left(\frac{c_n}{z-\alpha}a\right) \cdot \varphi(z-\alpha) = 0. \end{aligned}$$

The Lemma is proved.

**Corollary.** *Let $A$ be a closed subalgebra of $C(M)$ containing $H^\infty(\mathbb{D})$ with the property $M(A) = M$. Let $\mathscr{E}$ be its essential set. Then the following conditions are equivalent*:

(1) $\mathscr{E} \subset M \backslash \mathbb{D}$;

(2) $C(\overline{\mathbb{D}}) \subset A$.

The following examples (the first is due to Ivanov [7]) show that $M(A)$ can be very different from $A$.

**Example 1.** $A_1 = \mathrm{alg}[H^\infty, C_0(\mathbb{D})]$, where $C_0(\mathbb{D})$ denotes compactly supported in $\mathbb{D}$ continuous functions.

**Example 2.** $A_2 = \{f \in C(M) \colon f|M\backslash\mathbb{D} \in H^\infty|M\backslash\mathbb{D}\}$.

In both examples for $f \in A_i$ we can consider $\check{f}$ which is "boundary values of $f$ on $\partial\mathbb{D}$". In Example 1, this is straightforward. In Example 2, given $f$, find $F \in H^\infty$ such that $f|M\backslash\mathbb{D} = F|M\backslash\mathbb{D}$ and $\check{f} = F$ on $\mathbb{T}$. Operator $f \to \check{f}$ is bounded and multiplicative. Now for $z_0 \in \mathbb{D}$ the functional

$$\check{z}_0(f) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_0|^2}{|1-e^{-i\theta}z_0|^2}\check{f}(e^{i\theta})\,d\theta$$

gives a homomorphism, $\check{z}_0 \in P^{-1}(z_0)$, $\check{z}_0 \neq z_0$.

The following examples show that $M(A) = M$ is by no means sufficient for the inclusion $C(\overline{\mathbb{D}}) \subset A$.

**Example 3.** Take $A = \mathrm{alg}[H^\infty, C_A(\mathbb{D}_{1/2})]$, where $C_A(\mathbb{D}_{1/2}) = C(\mathbb{D}) \cap \mathrm{Hol}(\mathbb{D}_{1/2})$, $\mathbb{D}_{1/2} \stackrel{\text{def}}{=} \{|z| < \frac{1}{2}\}$. Then $H^\infty(\mathbb{D}) \subsetneqq A \subset C(M)$ but $C(\mathbb{D}) \not\subset A$. Let us show that $M(A) = M$. Let $\varphi \in X \equiv M(A)$ and $\varphi(z) = \alpha \in \mathbb{T}$. Let $\mu_\varphi$ be a representing measure on $M$ for $\varphi$. If $M_\alpha$ denotes the fiber of $M$ over $\alpha$ then clearly $\operatorname{supp}\mu_\varphi \subset M_\alpha$. Now let $\varphi_1, \varphi_2 \in X$, $P\varphi_1 = P\varphi_2 = \psi \in M$ and $\psi \in M_\alpha$. Then for any $a \in A$ we will have $(h \in H^\infty)$

$$\varphi_1(a) = \int_{M_\alpha} a\,d\mu_{\varphi_1} = \int_{M_\alpha} h\,d\mu_{\varphi_1} = \varphi_1(h) = \psi(h).$$

The second equality follows because $A|M_\alpha = H^\infty|M_\alpha$ for any $\alpha \in \mathbb{T}$. The same can be written for $\varphi_2$. Thus $\varphi_1 = \varphi_2$.

Now let $\varphi(z) = z_0 \in \mathbb{D}_{1/2}$. Let $a \in C_A(\mathbb{D}_{1/2+\varepsilon})$. Then

$$\varphi(a) = \varphi\left(\frac{a - a(z_0)}{z - z_0}\cdot(z - z_0)\right) + a(z_0) = a(z_0).$$

But $\bigcup_{\varepsilon>0} C_A(\mathbb{D}_{1/2+\varepsilon})$ is dense in $C_A(\mathbb{D}_{1/2})$ and so $\varphi(a) = a(z_0)$, $a \in C_A(\mathbb{D}_{1/2}) \cup H^\infty$. Thus $P^{-1}z_0 = z_0$, $z_0 \in \mathbb{D}$.

The next example shows that $M(A) = M$ is by no means necessary for the inclusion $C(\overline{\mathbb{D}}) \subset A$.

**Example 4.** $A = \mathrm{alg}[H^\infty, C(\overline{\mathbb{D}}), C_0(\lambda)]$, where $C_0(\lambda)$ denotes the following algebra of functions on $\mathbb{D}$. Given a sequence $\lambda = \{\lambda_n\}$ of points, $\lambda_n \to 1$, $0 \le \lambda_n < 1$, $\lambda_0 = 0$,

$$\frac{1-\lambda_{n+1}}{1-\lambda_n} \to 0$$

and discs $D_n = \{z: |\frac{z-\lambda_n}{1-\bar{\lambda}_n z}| \le \frac{1}{2}\}$, $C_0(\lambda) = \{f$ on $\mathbb{D}$: $f|D_0 \in C_0(D_0)$ and $f(\frac{z-\lambda_n}{1-\bar{\lambda}_n z}) = f(z)$, $z \in D_n$, $n = 1, 2, \dots$ ; $f \equiv 0$ on $\mathbb{D}\setminus\bigcup_{n\ge 0} D_n\}$. It is easy to check that $A \subset C(M)$. Let us explain why $M(A) \ne M$. Let $m \in \operatorname{clos}\{\lambda_n\}$ in $M$, and let $P_m$ be its Gleason part in $M$. In $M(A)$ we have at least two homomorphisms above some $t \in P_m$. In fact, one is $t$ itself. Another is given by the formula

$$\varphi_t(f) = \int_{S_t} f(s)\, d\mu_t(s), \qquad f \in A,$$

where $\mu_t$ is a representing measure for $t$ with support in $M(L^\infty) \subset M$. Clearly $\varphi_t \ne t$ as $\varphi_t(f) = 0$ for any $f \in C_0(\lambda)$.

## 3. The algebra generated by $H^\infty(\mathbb{D})$ and a harmonic function

In this section we will provide a natural proof of a remarkable theorem due to Axler and Shields, which is the main result in [1]. Also [2] contains another proof of this result.

**Theorem 2.** *Let $f$ be a bounded function that is harmonic but not analytic on $\mathbb{D}$. Then $H^\infty(\mathbb{D})[f]$ contains $C(\overline{\mathbb{D}})$.*

*Proof.* The proof will be divided into two steps.

*Step* 1. $M(H^\infty(\mathbb{D})[f]) = M$.

To prove Step 1, let $f = u + iv$, and let $u^*$ be the harmonic conjugate of $u$ so that $h = u + iu^*$ is analytic on $\mathbb{D}$. Let $\varphi_{|H^\infty(\mathbb{D})} = \psi \in M$. By [8, Theorem 21, p. 76], $\varphi$ has a representing measure $m$ on $M$, which is also an Arens-Singer measure for $\varphi$. Thus $\log|\varphi(e^h)| = \int \log|e^h|\, dm = \int u\, dm$. On the other hand, $\log|\varphi(e^h)| = \log|\psi(e^h)| = u(\psi)$, where the last equality follows from the proof of Lemma 4.4 in [6]. Consequently, $u(\psi) = \int u\, dm$. Similarly, $v(\psi) = \int v\, dm$ and hence $f(\varphi) = f(\psi)$, completing the proof of Step 1.

*Step* 2. The essential set of $H^\infty(\mathbb{D})[f]$ is contained in $M\setminus\mathbb{D}$.

Let $E$ be a nontrivial maximal antisymmetric set for $H^\infty(\mathbb{D})[f]$. Following Axler and Shields [1], we let $F(z) = e^f e^{-(u+iu^*)}$ and $G(z) = e^{if} e^{v+iv^*}$. Then $F$ and $G$ are constants on $E$.

If $E \cap \mathbb{D} \ne \varnothing$ then on $E \cap \mathbb{D}$ we have:

(1) $f = u + iu^* + c_1$, and

(2) $if = -(v + iv^*) + c_2$, where $c_1$ and $c_2$ are constants.

From (1) and (2) we get: $u + iu^* = i(v + iv^*) + k$ on $E \cap \mathbb{D}$, where $k$ is some constant. The function $g(z) = u + iu^* - i(v + iv^*) - k$ is analytic on $\mathbb{D}$, and moreover $g$ is a nonconstant function on $\mathbb{D}$, otherwise this would imply that $f$ is analytic on $\mathbb{D}$. Thus the zeros of $g$ are at most countable and consequently $E \cap \mathbb{D}$ is at most countable.

By Step 1, and the fact that $E$ is a weak peak set for $H^\infty(\mathbb{D})[f]$, we get $M(H^\infty(\mathbb{D})[f]_{|E}) = E$, and so by the Shilov idempotent theorem [8, p. 112] we get $E$ is connected. Hence $E \cap \mathbb{D} = \{\alpha\}$. Because $E$ is nontrivial, $(M\setminus\mathbb{D}) \cap E \ne \varnothing$. Let $u_\alpha$ be a closed set containing $\alpha$ and such that $U_\alpha \cap (E \cap (M\setminus\mathbb{D})) = \varnothing$. Hence $E = (u_\alpha \cap E) \cup (M\setminus\mathbb{D}) \cap E$, which is impossible because $E$ is connected. This contradiction shows that $E \subset M\setminus\mathbb{D}$, and consequently the essential set for $H^\infty(\mathbb{D})[f]$ is contained in $M\setminus\mathbb{D}$ completing the proof of Step 2.

Finally, Steps 1 and 2 and the conclusion of Theorem 1 completes the proof of Theorem 2.

*Remark* 1. That $M(H^\infty(\mathbb{D})[f]) = M$ if $f$ is a harmonic nonholomorphic function was proved by O. Ivanov in [7]. We are grateful to the referee for pointing this out to us and also for a better form of Theorem 1.

*Remark* 2. Theorems 1 and 2 can be proved for finitely connected domains exactly along the same lines. It is known (but the proof is quite difficult) that Theorem 2 holds for Widom domains [2].

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