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UNIFORM HOMEOMORPHISMS BETWEEN THE UNIT BALLS IN L_p AND l_p

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ABSTRACT. Let $T: B(L_p) \to B(l_p)$, $1 \le p < 2$, be a uniform homeomorphism with modulus of continuity δ_T. It is shown that for any γ, $0 \le \gamma < \frac{2-p}{2p}$, there exists $K > 0$ and a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that $\delta_T^{-1}(\delta_T(\varepsilon_n)) \ge K\varepsilon_n |\log \varepsilon_n|^\gamma$ for all ε_n.

It was proved by Mazur [6] that for p, $q \ge 1$ the spaces L_p and L_q, l_q are homeomorphic. From this work it also follows that the unit balls $B(L_p)$ and $B(L_q)$, $B(l_q)$ are uniformly homeomorphic. However, in Lindenstrauss [4] and Enflo [2] the nonexistence of a uniform homeomorphism between L_p and L_q was established. Enflo also proved that L_1 and l_1 are not uniformly homeomorphic (see [3, pp. 30–32]), and in [1] this result is generalized by Bourgain to the case $1 \le p < 2$. From the argument used in [1] and [3] it also follows that the unit balls in L_p and l_p are not Lipschitz equivalent. This was also known before (see [3, p. 27]).

In order to get more quantitative information about uniform homeomorphisms between L_p and l_p we study the *modulus of continuity*, $\delta_T(\varepsilon)$, defined by

$$\delta_T(\varepsilon) = \sup\{\|T(x_1) - T(x_2)\| : \|x_1 - x_2\| \le \varepsilon\}.$$

In [5] it was proved by the author that for $p = 1$ there is a sequence (ε_n) with $\varepsilon_n \to 0$ and a $K > 0$ such that

$$(*) \qquad \delta_{T^{-1}}(\delta_T(\varphi_n)) \ge K\varepsilon_n |\log \varepsilon_n| \quad \text{for all } \varepsilon_n.$$

In this paper we will give a similar result for the case $1 \le p < 2$. More precisely, we have the following.

Theorem 1. *Let $T: B(L_p) \to B(l_p)$, $1 \le p < 2$, be a uniform homeomorphism, and let γ be any number satisfying $0 \le \gamma < \frac{2-p}{2p}$. Then there exist a $K > 0$ and a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that*

$$(**) \qquad \delta_{T^{-1}}(\delta_T(\varepsilon_n)) \ge K\varepsilon_n |\log \varepsilon_n|^\gamma \quad \textit{for all } \varepsilon_n.$$

The main idea of the proof of $(*)$ was to construct a noncompact set of well-separated metric midpoints in L_1. These points are mapped on "almost

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metric midpoints" in l_1, and by a simple compactness argument we can find well-separated points in L_1 such that the distance between their images is much smaller. To prove Theorem 1 we just modify the ideas used in proof of $(*)$. Instead of using well-separated metric midpoints we use well-separated "almost metric midpoints" in L_p. For the proof of Theorem 1 we need the following lemma. Using our method we get the logarithmic estimate. We do not know what is the best estimate.

Lemma 1. *Let* α, γ *be any numbers with* $0 \leq \alpha$, $0 \leq \gamma \leq 1$. *If* $\lim_{\varepsilon \to 0} \frac{\delta_T(\varepsilon)}{\varepsilon|\log \varepsilon|^\gamma} = 0$, *then there exists a sequence* $\{\varepsilon_n\}$ *with* $\varepsilon_n \to 0$ *such that*

$$\delta_T\left(\frac{\varepsilon_n}{2}\sqrt{1+|\log \varepsilon_n|^{-2\alpha}}\right) \leq \frac{1}{2}\sqrt{1+|\log \varepsilon_n|^{-2\alpha}}(1+\log 2|\log \varepsilon_n|^{-1})\delta_T(\varepsilon)$$

for all ε_n.

In the proofs of Theorem 1 and Lemma 1 the following properties of $\delta_T(\varepsilon)$ will be used frequently. The proofs are simple and will be omitted.

(a) There exists a $K > 0$ such that $\delta_T(\varepsilon) \geq K\varepsilon$ for all ε.
(b) For every integer N we have $\delta_T(\frac{\varepsilon}{N}) \geq \frac{1}{N}\delta_T(\varepsilon)$.

Proof of Lemma 1. Let $\delta_T(\varepsilon) = K(\varepsilon)\varepsilon|\log \varepsilon|^\gamma$. Then there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that

$$\text{(I)} \qquad K\left(\frac{\varepsilon_n}{2}\sqrt{1+|\log \varepsilon_n|^{-2\alpha}}\right) \leq K(\varepsilon_n).$$

To see this we assume the contrary. Then there exists ε_0 such that for all ε, $0 < \varepsilon \leq \varepsilon_0$, we have $K(\varepsilon) < K\left(\frac{\varepsilon}{2}\sqrt{1+|\log \varepsilon_n|^{-2\alpha}}\right)$. Let $0 < \varepsilon_1 \leq \varepsilon_0$, and let $\varepsilon_n = \frac{\varepsilon_{n-1}}{2}\sqrt{1+|\log \varepsilon_{n-1}|^{-2\alpha}}$, $n = 1, 2, 3 \ldots$. Then we get $0 < K(\varepsilon_1) < K(\varepsilon_2) < K(\varepsilon_3) < \cdots < K(\varepsilon_N)$. This gives a contradiction since, by assumption, $K(\varepsilon_n) \to 0$ when $N \to \infty$. Now let $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, be a sequence satisfying (I) and let $s = \sqrt{1+|\log \varepsilon_n|^{-2\alpha}}$. Since $\gamma \leq 1$, we get $(|\log \varepsilon_n| + \log 2)^\gamma \leq |\log \varepsilon_n|^\gamma(1+\log 2|\log \varepsilon_n|^{-1})$. From this inequality and by (I) we obtain for ε_n small enough

$$\begin{aligned}\delta_T(\varepsilon_n s_n/2) &\leq \varepsilon_n s_n K(\varepsilon_n)|\log \varepsilon_n s_n/2|^\gamma/2 \leq \varepsilon_n s_n K(\varepsilon_n)|\log \varepsilon_n/2|^\gamma/2 \\ &\leq \delta_T(\varepsilon_n)s_n(1+\log 2|\log \varepsilon_n|^{-1})\end{aligned}$$

and the lemma is proved.

Proof of Theorem 1. Since L_p contains l_2 isometrically, it is enough to prove the theorem for T a uniform homeomorphism from $B(l_2)$ into $B(l_p)$. Given γ, $0 \leq \gamma < \frac{2-p}{2p}$, we let α be any number satisfying $\frac{p}{2-p}\gamma < \alpha < \frac{1}{p} - \gamma$.

We first assume that for some $K_1 > 0$ there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ such that $\delta_T(\varepsilon_n) \geq K_1\varepsilon_n|\log \varepsilon_n|^\gamma$ for all ε_n. Since we always can find $K_2 > 0$ such that $\delta_{T^{-1}}(\varepsilon) \geq K_2\varepsilon$ for all $\varepsilon > 0$, the theorem follows trivially for this case. Now, if we cannot find such a sequence for any $K_1 > 0$, then we have

$$\lim_{\varepsilon \to 0} \frac{\delta_T(\varepsilon)}{\varepsilon|\log \varepsilon|^\gamma} = 0.$$

Thus, by Lemma 1, we can find a sequence $\{\varepsilon_n\}$ satisfying the inequality (I).

Let $K_1 > 0$ and $K(\varepsilon)$ be such that $K_1\varepsilon \le \delta_T(\varepsilon) = K(\varepsilon)\varepsilon|\log\varepsilon|^\gamma\ \forall\varepsilon \le 1$. Let ε_n be in the sequence, and let r be such that

$$0 < r < 1 - \frac{1}{2}\sqrt{1 + |\log\varepsilon_n|^{-2\alpha}}(1 + \log 2|\log\varepsilon_n|^{-1}).$$

Then we can find $x = \sum x_i e_i$, $y = \sum y_i e_i$ in $B(l_2)$ supported by a finite number of coordinates and with $\|x - y\| < \varepsilon_n$ such that

$$\|T(x) - T(y)\| > (1 - r)\delta_T(\varepsilon_n).$$

By Lemma 1 and by the assumption of r this inequality implies that $\|x - y\| \ge \varepsilon_n/2$. For any i outside the union of the supports of x and y we let z_i be the almost metric midpoint to x, y defined by

$$z_i = \frac{x+y}{2} + \|x - y\||\log\|x - y\||^{-2\alpha}e_i.$$

Then we have

$$\|z_i - x\| = \|z_i - y\| = \frac{\|x-y\|}{2}\sqrt{1 + |\log\|x - y\||^{-2\alpha}} \le \frac{\varepsilon_n}{2}\sqrt{1 + |\log\varepsilon_n|^{-2\alpha}},$$

$$\|z_i - z_i\| = \frac{1}{\sqrt{2}}\|x - y\||\log\|x - y\||^{-\alpha} \ge \frac{1}{\sqrt{2}}\frac{\varepsilon_n}{2}\left|\log\frac{\varepsilon_n}{2}\right|^{-\alpha},$$

$$\|T(x) - T(z_i)\| \le \delta_T\left(\frac{\varepsilon_n}{2}\sqrt{1 + |\log\varepsilon_n|^{-2\alpha}}\right),$$

and

$$\|(Ty) - T(z_i)\| \le \delta_T\left(\frac{\varepsilon_n}{2}\sqrt{1 + |\log\varepsilon_n|^{-2\alpha}}\right).$$

Now, let A be a finite set such that up to a neglible error $T(x)$ and $T(y)$ are supported on A. We let π denote the coordinate projection. Then we have

$$(1 - r)\delta_T(\varepsilon_n) < \|T(x) - T(y)\| \le \|T(x) - \pi_A T(z_i)\| + \|T(y) - \pi_A T(z_i)\|$$

and

$$\begin{aligned}\|T(x) - T(z_i)\|^p &= \|\pi_A T(x) - \pi_A T(z_i)\|^p + \|\pi_{N\setminus A}T(x) - \pi_{N\setminus A}T(z_i)\|^p \\ &\ge \|T(x) - \pi_A T(z_i)\|^p + \|\pi_{N\setminus A}T(z_i)\|^p - r.\end{aligned}$$

Similarly for x replaced by y. Thus we get

$$\text{(i)}\quad \begin{aligned}2\left(\delta_T\left(\frac{\varepsilon_n}{2}\sqrt{1 + |\log\varepsilon_n|^{-2\alpha}}\right)\right)^p &\ge \|T(x) - T(z_i)\|^p + \|T(y) - T(z_i)\|^p \\ &\ge \|T(x) - \pi_A T(z_i)\|^p + \|T(y) - \pi_A T(z_i)\|^p + 2\|\pi_{N\setminus A}T(z_i)\|^p - 2r \\ &\ge 2^{-p/p'}(1 - r)^p(\delta_T(\varepsilon_n))^p + 2\|\pi_{N\setminus A}T(z_i)\|^p - 2r.\end{aligned}$$

Let $C_n = \inf_{i,j;\ i\not\equiv j}\|T(z_i) - T(z_j)\|$. Then we have the following.

Claim.

$$\lim_{n\to\infty}\frac{C_n}{\delta_T(\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)})} = 0.$$

To prove the claim we assume the contrary, i.e., there exists $K > 0$ and an infinite subsequence of $\{\varepsilon_n\}$ such that

$$C_n \ge K\delta_T(\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)}).$$

Thus we have

(ii) $$\|T(z_i) - T(z_j)\| \geq K\delta_T(\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)})$$

for all i, j and for all ε_n in the subsequence.

Since A is finite, we have that $\overline{\{\pi_A T(z_i)\}}$ is compact. Thus by (ii) and the triangle inequality we get unit vectors e_i for which

$$\|\pi_{N\setminus A}T(z_i)\| \geq \frac{1}{2}K\delta_T(\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)}).$$

This together with (i) gives

$$2\left(\delta_T\left(\frac{\varepsilon_n}{2}\sqrt{1+|\log\varepsilon_n|^{-2\alpha}}\right)\right)^p - 2^{-p/p'}(1-r)^p(\delta_T(\varepsilon_n))^p \geq 2^{1-p}K^p\delta_T(\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)})^p - 4r.$$

Since this holds for r arbitrarily small, we get

$$2\left(\delta_T\left(\frac{\varepsilon_n}{2}\sqrt{1+|\log\varepsilon_n|^{-2\alpha}}\right)\right)^p - 2^{-p/p'}(\delta_T(\varepsilon_n))^p \geq 2^{1-p}K^p\delta_T(\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)})^p.$$

Using the fact that

$$\delta_T(\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)}) \geq \frac{1}{2}|\log\varepsilon_n|^{-(\alpha+\gamma)}\delta_T(\varepsilon_n)$$

and Lemma 1 we obtain

$$2^{1-p}\left(\sqrt{1+|\log\varepsilon_n|^{-2\alpha}}(1+\log 2|\log\varepsilon_n|^{-1})\right)^p - 2^{-2/2'} \geq 2^{1-2p}K^p|\log\varepsilon_n|^{-p(\alpha+\gamma)},$$

so we have

$$|\log\varepsilon_n|^{p(\alpha+\gamma)}\left(\left(\sqrt{1+|\log\varepsilon_n|^{-2\alpha}}(1+\log 2|\log\varepsilon_n|^{-1})\right)^p - 1\right) \geq 2^{-p}K^p.$$

Since by assumption of α, γ we have $p(\alpha+\gamma)-2\alpha < 0$ and $p(\alpha+\gamma)-1<0$, this inequality gives a contradiction for ε_n small enough. Thus we have proved that

$$\lim_{n\to\infty}\frac{C_n}{\delta_T(\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)})} = 0.$$

Now assume that for some $\varepsilon_0 > 0$ and $K>0$ we have

$$\delta_{T^{-1}}(\delta_T(\varepsilon_n)) \leq K\varepsilon_n|\log\varepsilon_n|^{\gamma} \quad \text{for all } \varepsilon_n < \varepsilon_0.$$

Given integer N there exists $\varepsilon' < \varepsilon_0$ such that for all $\varepsilon_n < \varepsilon'$ we can find i, j such that

$$\|T(z_i) - T(z_j)\| < \frac{1}{N}\delta_T(\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)}).$$

Thus we have

$$\begin{aligned}
\frac{1}{8}\varepsilon_n|\log\varepsilon_n|^{-\alpha} &< \frac{1}{\sqrt{2}}\|x-y\||\log\|x-y\||^{-\alpha} = \|z_i - z_j\| \\
&\leq \delta_{T^{-1}}\left(\frac{1}{N}\delta_T(\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)})\right) \leq \delta_{T^{-1}}\left(\delta_T\left(\frac{1}{N}\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)}\right)\right) \\
&\leq K\frac{1}{N}\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)}\left|\log\varepsilon_n + \log\frac{1}{N}|\log\varepsilon_n|^{-(\alpha+\gamma)}\right|^{\gamma} \\
&\leq K\frac{2}{N}\varepsilon_n|\log\varepsilon_n|^{-(\alpha+\gamma)+\gamma} \quad \text{for } \varepsilon_n \text{ small enough}.
\end{aligned}$$

From this we get $N \leq 16K$, and by taking N large enough we get a contradiction. Thus there exists $K > 0$ and an infinite subsequence

$$\left\{\varepsilon_n' ; \varepsilon_n' \frac{1}{N}\varepsilon_n |\log \varepsilon_n|^{-(\alpha+\gamma)}\right\}$$

such that $\delta_{T^{-1}}(\delta_T(\varepsilon_n)) \geq K\varepsilon_n |\log \varepsilon_n|^{\gamma}$ and the theorem is proved.

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