PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 3, March 1995

DOLD MANIFOLDS WITH $(\mathbf{Z}_2)^k$-ACTION

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(Communicated by Thomas Goodwillie)

ABSTRACT. Actions on Dold manifolds $P(m, 2^k)$ are constructed to determine the ideals $\mathscr{I}_{*,k}^{2^k}$ of cobordism classes in $\mathscr{N}_*$ containing a representative admitting a $(\mathbf{Z}_2)^k$-action with fixed point set of codimension 2^k.

1. INTRODUCTION

Let $\mathscr{I}_{*,k}^{r} = \bigoplus_{n=r}^{\infty} \mathscr{I}_{n,k}^{r}$ denote the ideal in the unoriented cobordism ring $\mathscr{N}_*$ of classes containing a representative that admits a $(\mathbf{Z}_2)^k$-action with fixed point set of constant dimension $n-r$. In [Sh, 5.1] it is shown that for $1 < r < 2^k$

$$\mathscr{I}_{*,k}^{r} = \begin{cases} \bigoplus_{n=r}^{\infty} \mathscr{N}_n, & r \text{ even}, \\ \bigoplus_{n=r}^{\infty} \mathscr{N}_n \cap \operatorname{Ker} \chi, & r \text{ odd}, \end{cases}$$

where $\chi : \mathscr{N}_* \to \mathbf{Z}_2$ denotes the mod 2 Euler characteristics. This is a "best possible" result in that it follows from [tD, 1] that $\mathscr{I}_{n,k}^{r}$ contains no indecomposables if $n - r < [n/2^k]$ and so $\mathscr{I}_{2^k,k}^{2^k} \neq \mathscr{N}_{2^k}$.

In general the determination of $\mathscr{I}_{*,k}^{r}$ requires methods particular to the choice of r, k (see [P]). As one might suspect from [tD, 1] it is usually hardest to construct examples on indecomposables in dimensions close to r. The present work proves the

Proposition. *For $k > 1$ the ideal $\mathscr{I}_{*,k}^{2^k}$ consists of all classes in dimensions greater than 2^k and the decomposables in dimension 2^k.*

Hence whereas $\mathscr{I}_{*,1}^{2}$ contains all classes $\alpha \in \bigoplus_{n=2}^{\infty} \mathscr{N}_n$ except those for which $w_1^n(\alpha) \neq 0$ [S, 9.2], for $k > 1$ $\mathscr{I}_{*,k}^{2^k}$ contains all classes $\alpha \in \bigoplus_{n=2^k}^{\infty} \mathscr{N}_n$ except those in $\mathscr{N}_{2^k}$ for which $s_{2^k}(\alpha) = w_1^{2^k}(\alpha) \neq 0$.

The author thanks Professor R. E. Stong for helpful conversations.

Received by the editors March 4, 1993 and, in revised form, June 8, 1993; presented in January 1993 at the AMS-MAA joint meeting in San Antonio, TX.

1991 *Mathematics Subject Classification.* Primary 57R85; Secondary 57S17.

Key words and phrases. Cobordism class, Dold manifold, fixed point set, line bundle, projective space bundle, $(\mathbf{Z}_2)^k$-action.

2. Background

The unoriented cobordism ring

$$\mathcal{N}_* = \bigoplus_{n=0}^{\infty} \mathcal{N}_n$$

is a $\mathbf{Z}_2$-polynomial algebra on a single generator x_i of each dimension not of the form $2^u - 1$ [T]. Manifolds that can be chosen as representatives of generators in their dimensions include the following *indecomposables*.

The *Dold manifold* $P(m, n)$ is the quotient space obtained from $S^m \times CP^n$ by identifying $[x, z]$ with $[-x, \bar{z}]$. It is indecomposable in $\mathcal{N}_*$ if and only if n is even and $\binom{n+m-1}{n} = (n+m-1)!/n!(m-1)!$ is odd [D, 3].

Let $RP(n_1, n_2, \dots, n_l)$ be the projective space bundle of $\lambda_1 \oplus \cdots \oplus \lambda_l$ over $RP^{n_1} \times \cdots \times RP^{n_l}$, where λ_i is the pullback of the usual line bundle over the ith factor. Let $n = \sum_{i=1}^{l} n_i$. For $l > 1$, the $(l+n-1)$-dimensional $RP(n_1, n_2, \dots, n_l)$ is indecomposable in $\mathcal{N}_*$ if and only if $\sum_{i=1}^{l} \binom{l+n-2}{n_i}$ is odd [S, 3.4].

For the calculation of binomial coefficients it is useful to recall Kummer's result stating that if

$$m = \sum_{i=0}^{k} m_i 2^i \quad \text{and} \quad n = \sum_{i=0}^{k} n_i 2^i$$

with $0 \le m_i, n_i \le 1$, then $\binom{m}{n}$ is odd if and only if $m_i \ge n_i$ for every i.

3. Main proof

The primary task is to exhibit indecomposable manifolds in all possible dimensions. Then previous results lead to all positive dimension decomposables.

Proof of Proposition. As previously stated the result [tD, 1] precludes the existence in $\mathcal{I}_{*,k}^{2^k}$ of an indecomposable of dimension 2^k. For $2^k < d$, d not of the form $2^u - 1$, indecomposables may be found as follows.

Dimensions $2^k + 1 \le d \le 2^k + 2^{k-1}$. Let $CP^{2^{k-1}}$ be expressed in homogeneous coordinates as $[z_1, z_2, \dots, z_{2^{k-1}}, z_{2^{k-1}+1}]$. For $1 \le j \le k$ define T_j to act as multiplication by -1 on the coordinate z_i if $i = 1, 2, 3, \dots, 2^{j-1} \bmod 2^j$ and to leave fixed the other coordinates. Then F_{T_k} consists of a copy of $CP^{2^{k-1}-1}$ along with a point (CP^0), $F_{T_k, T_{k-1}}$ is two copies of $CP^{2^{k-2}-1}$ with the copy of $CP^0, \dots,$ and finally $F_{T_k, T_{k-1}, \dots, T_1}$ is a set of $2^{k-1}+1$ points.

This induces on each $(m+2^k)$-dimensional Dold manifold $P(m, 2^{k-1})$ an action that fixes $2^{k-1}+1$ copies of S^m, i.e., has a fixed point set of codimension 2^k. For $1 \le m \le 2^{k-1}$ the $(k-1)$th digit of the dyadic expansion of $2^{k-1} + m - 1$ is 1, so

$$\binom{2^{k-1}+m-1}{2^{k-1}}$$

is odd and therefore $P(m, 2^{k-1})$ is indecomposable.

Dimensions $2^k + 2^{k-1} + 1 \le d \le 2^{k+1} - 2$. An analogous argument constructs an appropriate action on $RP(\bigoplus_{i=1}^{2^{k-1}+1} \xi_i)$. Let T_j act as multiplication by -1

in ξ_i if $i = 1, 2, 3, \dots, 2^{j-1} \bmod 2^j$. Then

$$F_{T_k} = RP\left(\bigoplus_{i=1}^{2^{k-1}} \xi_i\right) \cup RP(\xi_{2^{k-1}+1}),$$

$$F_{T_k, T_{k-1}} = RP\left(\bigoplus_{i=1}^{2^{k-2}} \xi_i\right) \cup RP\left(\bigoplus_{i=2^{k-2}+1}^{2^{k-1}} \xi_i\right) \cup RP(\xi_{2^{k-1}+1}),$$

$$\vdots$$

$$F_{T_k, T_{k-1}, \dots, T_1} = \bigcup_{i=1}^{2^{k-1}+1} RP(\xi_i).$$

For $RP(n_1, n_2, \dots, n_{2^k+2})$ with each $\xi_i^2 = \lambda_{2i-1} \bigoplus \lambda_{2i}$ this is a $(\mathbf{Z}_2)^k$-action on a (2^k+1)-dimensional projective bundle with fixed point set a union of 1-dimensional bundles.

Since

$$\binom{2^k + 2^{k-1}}{p} = 0 \quad \text{for } 2 \le p \le 2^{k-1} - 1,$$

application of the familiar identity $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ allows us to conclude that

$$\binom{2^k + 2^{k-1} + (p-2)}{p} = 0 \quad \text{for } 2 \le p \le 2^{k-1} - 1.$$

Let $m = 2^k + 2^{k-1} + p - 2$. Then the $(2^k + 2^{k-1} + p - 1)$-manifold $RP(p, 2^{k-2} - 1, 2^{k-2} - 1, n_4 = 0, 0, \dots, n_{2^k+2} = 0)$ is indecomposable since

$$\binom{m}{p} + 2\binom{m}{2^{k-2} - 1} + (2^k - 1)\binom{m}{0} = 0 + 0 + 1 = 1 \mod 2.$$

Dimensions $2^{k+1} \le d$, $d \ne 2^u - 1$. By [Sh, 4.1] there exist indecomposable manifolds $RP(n_1, \dots, n_{2^k+2})$ in all such dimensions except the powers of two. For those dimensions consider the following special case of [Sh, 3.1].

Let T be an involution on RP^{2r+1} fixing two copies of RP^r. Define T_1, T_2 on $RP(2r+1, p, q)$; thus, T_1 is induced by the action on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ over $RP^{2r+1} \times RP^p \times RP^q$ that consists of multiplication by -1 in λ_1, and T_2 by the action that extends T linearly on λ_1 and acts as -1 in λ_2. Then $F_{T_1} = RP(\lambda_1) \cup RP(\lambda_2 \oplus \lambda_3)$. Since the former component is just a copy of the base, one can write F_{T_1} as $RP^{2r+1} \times RP(\lambda_1) \cup RP(\lambda_2 \oplus \lambda_3)$ understanding that all bundles are now over the base $RP^p \times RP^q$. Then it is apparent that F_{T_1, T_2} consists of six copies of $RP^r \times RP^p \times RP^q$. Specifically, if $r = 2^k - 3$, $q = 0$, the $(2^{k+1} - 3 + p)$-manifold $RP(2^{k+1} - 5, p, 0)$ has fixed point set six copies of $RP^{2^k - 3} \times RP^p$. Then p can easily be chosen to generate a manifold of dimension any appropriate power of 2.

If d is a power of 2, then $\binom{d-1}{i}$ is odd for all i and any $RP(2r+1, p, q)$ of that dimension is indecomposable.

Decomposables. Since $\mathscr{I}_{*,k}^{2^k}$ is an ideal, to complete the proof it is necessary only to show that it contains the class $\alpha\beta$ for $0 \ne \alpha \in \mathscr{N}_m$, $0 \ne \beta \in \mathscr{N}_n$, with $2 \le m \le n \le 2^k \le m + n$. For this it suffices to show $\alpha \in \mathscr{I}_{*,k}^r$, $\beta \in \mathscr{I}_{*,k}^{2^k - r}$ for some r and to consider the product action on representative manifolds with action.

From [Sh, 5.1] it follows that $\alpha \in \mathcal{I}^r_{*,k}$ for even $r < \min(m, 2^k - 1)$ and furthermore, if $\alpha \in \operatorname{Ker}\chi$ (which holds if m odd), then $\alpha \in \mathcal{I}^r_{*,k}$ for all $r < \min(m, 2^k - 1)$. A similar statement holds for β, n.

Therefore, if $m + n = 2^k$, take $\alpha \in \mathcal{I}^m_{*,k}$, $\beta \in \mathcal{I}^n_{*,k} = \mathcal{I}^{2^k-m}_{*,k}$. If $m + n = 2^k + 1$, then either m or n, say n, is odd, and $\alpha \in \mathcal{I}^m_{*,k}$, $\beta \in \mathcal{I}^{n-1}_{*,k} = \mathcal{I}^{2^k-m}_{*,k}$ suffice. If $m + n \geq 2^k + 2$, take $\alpha \in \mathcal{I}^{m-2}_{*,k}$, $\beta \in \mathcal{I}^{2^k-(m-2)}_{*,k}$. □

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