PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 3, March 1995

D-SETS AND BG-FUNCTORS IN KAZHDAN-LUSZTIG THEORY

YI MING ZOU

(Communicated by Roe Goodman)

ABSTRACT. By using Deodhar's combinatorial setting and Bernstein-Gelfand projective functors, this paper provides some necessary and sufficient conditions for a highest weight category to have a Kazhdan-Lusztig theory. A consequence of these conditions is that in the semisimple Lie algebra case, the Kazhdan-Lusztig conjecture on the multiplicities of a Verma module implies the nonnegativity conjecture on the coefficients of Kazhdan-Lusztig polynomials.

One of the central topics in representation theory in recent years is the so-called Kazhdan-Lusztig theory. The Kazhdan-Lusztig polynomials play a key role in this theory. These polynomials can be defined by using a distinguished basis of the Hecke algebra associated to a Coxeter group. In [KL1], there are two conjectures about these polynomials: (a) For any Coxeter group, the coefficients of these polynomials are nonnegative integers; (b) If the Coxeter group is the Weyl group of a complex semisimple Lie algebra, then the multiplicities of the composition series of a Verma module are given by the values of these polynomials at 1. Conjecture (b) is usually referred to as the Kazhdan-Lusztig conjecture and was proven in [BB] and [BK] shortly thereafter. Conjecture (a) is now known to be true for all crystallographic Coxeter groups (for a more up-to-date reference on recent developments of Kazhdan-Lusztig theory, we refer to [DS]). It was shown in [D] that if the coefficients of the Kazhdan-Lusztig polynomials of a Coxeter group are nonnegative, then these polynomials can be defined by using certain sets derived from the elements of the Coxeter group. In fact, these sets give a closed formula for the Kazhdan-Lusztig polynomials under the nonnegativity assumption (see [D]). Since the Kazhdan-Lusztig polynomials are not easy to get at in general, the results in [D] give strong evidence for the importance of the nonnegativitiness. In an attempt to understand the results of [D], we observed that in the semisimple Lie algebra case, conjecture (b) implies conjecture (a). The connection is provided by some tensor functors called projective functors defined in [BG]. In this paper, we will give some necessary and sufficient conditions for the validity of the Kazhdan-Lusztig conjecture in certain special cases of the highest weight categories defined by CPS (see [CPS1]

Received by the editors June 2, 1993; the contents of this paper have been presented to the Nineteenth Holiday Symposium held in December 1992 at New Mexico State University.
1991 *Mathematics Subject Classification.* Primary 22E47, 17B10; Secondary 22E46, 17B35.
Key words and phrases. D-sets, BG-functors.

and [CPS2]). In particular, our results will show that in the semisimple Lie algebra case, conjecture (b) implies conjecture (a).

This paper is arranged as follows: In section 1, we recall the definition of the highest weight categories, D-sets and BG-functors. In section 2, we discuss the relationship between D-sets and BG-functors. Some necessary and sufficient conditions for some special highest weight categories to have a Kazhdan-Lusztig theory (in the sense of [DS]) were given in section 3.

1. Definitions and notation

1.1. **Highest weight categories.** We recall the definition of highest weight categories given by CPS.

Let $\mathscr{C}$ be an abelian category over a field F. Then $\mathscr{C}$ is called locally artinian if it admits arbitrary union of its subobjects of finite length. In addition, we assume that $\mathscr{C}$ satisfies the Grothendieck condition: $B \cap (\cup A_\alpha) = \cup (B \cap A_\alpha)$ for a subobject B and a family of subobjects $\{A_\alpha\}$ of an object X. A composition factor S (also called a subquotient) of an object A in $\mathscr{C}$ is a composition factor of a subobject of finite length. The multiplicity of S in A, denoted $[A : S]$, is defined to be the supremum of the multiplicity of S in all subobjects of A of finite length. A poset Λ is said to be interval-finite provided that, for every $\mu \leq \lambda$ in Λ, the "interval" $[\mu, \lambda] = \{\tau \in \Lambda : \mu \leq \tau \leq \lambda\}$ is finite.

Definition (cf. [CPS1]). A locally artinian category $\mathscr{C}$ over F is called a highest weight category if there exists an interval-finite poset Λ (the "weights" of $\mathscr{C}$) satisfying the following conditions:

(a) There is a complete collection $\{L(\lambda)\}_{\lambda \in \Lambda}$ of nonisomorphic simple (irreducible) objects of $\mathscr{C}$ indexed by the set Λ.

(b) There is a collection $\{M(\lambda)\}_{\lambda \in \Lambda}$ of objects of $\mathscr{C}$ and, for each λ, a subobject $M'(\lambda) \subseteq M(\lambda)$ such that $M(\lambda)/M'(\lambda) \cong L(\lambda)$ and all composition factors $L(\mu)$ of $M'(\lambda)$ satisfy $\mu < \lambda$. For $\lambda, \mu \in \Lambda$, we have that $\dim_F \operatorname{Hom}_F(M(\lambda), M(\mu))$ and $[M(\lambda) : L(\mu)]$ are finite.

(c) Each simple object $L(\lambda)$ has a projective cover $P(\lambda)$ in $\mathscr{C}$. Also, each $P(\lambda)$ has a filtration

$$P(\lambda) = P^0 \supset P^1 \supset P^2 \supset \cdots \supset P^n \supset P^{n+1} = (0)$$

such that:

(i) $P^0/P^1 \cong M(\lambda)$,

(ii) for $k \geq 1$, $P^k/P^{k+1} \cong M(\mu_k)$ for some $\mu_k > \lambda$.

We call the objects $M(\lambda)$ ($\lambda \in \Lambda$) described in (b) Verma modules. For $\mu \in \Lambda$, we will use the notation $(P(\lambda) : M(\mu))$ to denote the number of i's in $[0, n]$ such that $P^i/P^{i+1} \cong M(\mu)$.

Remark. Our highest weight categories are actually the "duals" of some special highest weight categories defined in [CPS1]; the highest weight category of CPS is more general. For examples of highest weight categories, we refer to [CPS1] and [CPS2].

1.2. **Coxeter groups and Hecke algebras.** We further assume that there is a Coxeter group $\langle W, S \rangle$ (W is the group, S is the generating set, see [H] for the definition of a Coxeter group) associated to our highest weight category $\mathscr{C}$ such that W acts on Λ and satisfies the following conditions:

(i) for each $\lambda \in \Lambda$, there is a unique subgroup W_λ of W such that for any $x \in W_\lambda$, $[M(x \cdot \lambda), L(\mu)] \neq 0$ only if $\mu = y \cdot \lambda$ for some $y \in W_\lambda$,

(ii) if $\lambda_0 \in W_\lambda \cdot \lambda$ is a maximal element, then $x \cdot \lambda_0 \geq y \cdot \lambda_0 \Leftrightarrow x \leq y$ in the Chevalley order (known as Bruhat order before) of W.

The Coxeter groups which we will be interested in are just Weyl groups.

Let $\mathscr{H}$ be the Hecke algebra associated to $\langle W, S\rangle$; then $\mathscr{H}$ is a free $\mathbf{Z}[q^{1/2}, q^{-1/2}]$ module with basis $\{T_x : x \in W\}$ and multiplication defined by

$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w & \text{if } \ell(ws) < \ell(w), \end{cases}$$

where $w \in W$, $s \in S$, and ℓ is the length function of W. Recall that the following elements of $\mathscr{H}$

$$C'_y = q^{-\ell(y)/2} \sum_{x \leq y} P_{x,y} T_x, \qquad y \in W,$$

form a distinguished basis of $\mathscr{H}$, where $P_{x,y}$ are the Kazhdan-Lusztig polynomials.

1.3. **Duality condition.** Let $\mathscr{C}$ be a highest weight category. Then $\mathscr{C}$ is said to satisfy the duality condition provided that for $\lambda, \mu \in \Lambda$,

$$[M(\lambda) : L(\mu)] = (P(\mu) : M(\lambda)). \tag{1.3.1}$$

1.4. ***D*-sets.** We recall some notions introduced in [D].

Let $\langle W, S\rangle$ be a Coxeter group. Let e be the identity of W. For any $y \in W$ and a fixed reduced expression $y = s_1 s_2 \cdots s_k$, $s_i \in S$, $i = 1, \ldots, k$, we define $\mathscr{S}_y$ to be the set which consists all k-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ such that $x_i = s_i$ or e, $i = 1, 2, \ldots, k$. Thus, the cardinality of $\mathscr{S}_y = 2^k$. For our convenience, we set $\mathscr{S}_e = \{e\}$. We define the length of $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ to be the length of $x_1 x_2 \cdots x_k$, and we also use ℓ to denote the length of $\mathbf{x}$. Hence $\ell(\mathbf{x}) = \ell(x_1 x_2 \cdots x_k)$. For $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, let $\pi(\mathbf{x}) = x_1 x_2 \cdots x_k$. Define a map $d : \mathscr{S}_y \to \mathbf{Z}_+$ by

$$d(\mathbf{x}) = |\{j \in [2, k] : x_1 \cdots x_{j-1} s_j < x_1 \cdots x_{j-1}\}|.$$

A Coxeter group $\langle W, S\rangle$ is said to satisfy the D-condition, if for any $y \in W$ and a fixed reduced expression $y = s_1 \cdots s_k$, one can define (inductively on k) a subset $\delta(y)$ of $\mathscr{S}_y$ such that:

(1) $\delta(e) = e$.

(2) If for all $\mathbf{x} \in \mathscr{S}_y$,

$$d(\mathbf{x}) \leq (\ell(y) - \ell(\mathbf{x}) - 1)/2, \tag{1.4.1}$$

then $\delta(y) = \mathscr{S}_y$. Otherwise, there are subsets B_i of W, $1 \leq i \leq t$, consisting of elements $x \in W$ with $\ell(x) < \ell(y)$, such that one can find a reduced expression for each $x \in B_i$, $1 \leq i \leq t$, and the corresponding sets

$$\mathscr{B}_i = \bigcup_{x \in B_i} \delta(x)$$

and embeddings $\iota_i : \mathscr{B}_i \to \mathscr{S}_y$ such that:

(i) $\iota_i(\delta(x)) \cap \iota_i(\delta(x')) = \varnothing$ if $x \neq x'$, and $\iota_i(\mathscr{B}_i) \cap \iota_j(\mathscr{B}_j) = \varnothing$ if $i \neq j$;

(ii) if $\mathbf{x} \in \mathscr{S}_y$ and $d(\mathbf{x}) > (\ell(y) - \ell(\mathbf{x}) - 1)/2$, then $\mathbf{x} \in \iota_i(\mathscr{B}_i)$ for some $1 \leq i \leq t$;

(iii) $\delta(y) = \mathscr{S}_y - \bigcup_{i=1}^t \iota_i(\mathscr{B}_i)$.

(3) The following formula holds:

$$P_{x,y} = \sum_{\substack{\mathbf{x} \in \delta(y) \\ \pi(\mathbf{x}) = x}} q^{d(\mathbf{x})}.$$

Definition. If $\langle W, S \rangle$ satisfies the D-condition, then for each $y \in W$ and a fixed reduced expression $y = s_1 \cdots s_k$, the set $\delta(y)$ defined above is called a D-set.

Remarks. 1. Note that for each $y \in W$, one can have several D-sets (see [D, Section 4]).

2. It follows from the results in [D] that $\langle W, S \rangle$ satisfies the D-condition if and only if the corresponding Kazhdan-Lusztig polynomials have nonnegative coefficients. It is also known that all crystallographic Coxeter groups satisfy the D-condition.

Suppose that $\langle W, S \rangle$ satisfies the D-condition. Let $y \in W$. Fix a reduced expression of y and a D-set $\delta(y)$ corresponding to this reduced expression. Denote by $m_x(\delta(y))$ the number of appearances of an element x in the B_i's, and let

$$B = \bigcup_{i=1}^{t} B_i. \tag{1.4.2}$$

1.5. **Bernstein-Gelfand projective functors** (cf. [BG]). Let $\mathscr{C}$ be a highest weight category. Suppose that there is a Coxeter group $\langle W, S \rangle$ acting on the poset Λ and satisfying the assumptions in 1.2. Suppose further that the duality condition (1.3.1) is also satisfied.

Definition. An indecomposable functor $F : \mathscr{C} \to \mathscr{C}$ is called an indecomposable BG-functor, if:

(i) F is exact.

(ii) There is a $\lambda \in \Lambda$ such that if λ' is the maximal element of $W_\lambda(\lambda)$, then $F(M(\lambda')) = P(\lambda'')$ for some $\lambda'' \in W_\lambda(\lambda)$.

(iii) For any $\mu \notin W_\lambda(\lambda)$, $F(M(\mu)) = 0$.

(iv) If F^K is the operator induced by F on the Grothendieck group of $\mathscr{C}$, then F^K commutes with the W-action.

A functor $F : \mathscr{C} \to \mathscr{C}$ is called a BG-functor if F is the direct sum of indecomposable BG-functors.

We assume that the composition $F_1 \circ F_2$ of two BG-functors is again a BG-functor.

We say that $\mathscr{C}$ (with the action of a Coxeter group $\langle W, S \rangle$) has enough BG-functors provided that for any $P(\lambda)$, $\lambda \in \Lambda$, there is an indecomposable BG-functor F_λ and an object $M(\mu)$ (see Definition 1.1(b)) of $\mathscr{C}$ such that $F_\lambda(M(\mu)) = P(\lambda)$.

It is known from the results of [BG] (see Theorems 3.3 and 3.4 in [BG]) that for the highest weight category of a complex semisimple Lie algebra there are enough BG-functors. In [BG], these functors were defined by taking tensor products in the category with finite-dimensional objects in the same category.

2. D-SETS AND PROJECTIVE OBJECTS

In this section, we assume that we have a highest weight category $\mathscr{C}$ with a Coxeter group $\langle W, S\rangle$ acting on the poset Λ such that:

(1) There exists at least one $\lambda \in \Lambda$ such that for any $x \neq y \in W$, we have $x \cdot \lambda \neq y \cdot \lambda$.

(2) For any $x \in W$, $[M(x \cdot \lambda) : L(\mu)] \neq 0$ iff $\mu = y \cdot \lambda$ for some $y \geq x$.

(3) The duality condition (1.3.1) holds.

Proposition 1. *Assume that $\mathscr{C}$ has enough BG-functors. Then for each $y \in W$ and a reduced expression $y = s_1 \cdots s_k$, there is a BG-functor G_y such that:*

(i) *$G_y(M(\lambda)) = D_y$ is a projective object of $\mathscr{C}$.*

(ii) *$[D_y : M(\mu)] \neq 0$ iff $\mu = x \cdot \lambda$ for some $x \in W$ such that $e \leq x \leq y$.*

(iii) *Let $\mathscr{S}_y$ be the set corresponding to $y = s_1 \cdots s_k$ defined in* 1.4. *For any x such that $e \leq x \leq y$, let $D_y(x)$ be the number of elements $\mathbf{x} = (x_1, \ldots, x_k) \in \mathscr{S}_y$ such that $x_1 \cdots x_k = x$. Then $[D_y : M(\mu)] = D_y(x)$.*

Remark. From this proposition, we see that D_y corresponds to $\mathscr{S}_y$. Also note that since $\mathscr{S}_y$ depends on the chosen reduced expression of y, D_y depends on the chosen reduced expression of y.

Proof. Under the assumption of the proposition, there is an indecomposable BG-functor F_x for each $x \in W$ such that

$$F_x(M(\lambda)) = P(x \cdot \lambda),$$

where $P(x \cdot \lambda)$ is the projective cover of $L(x \cdot \lambda)$ (hence $P(x \cdot \lambda)$ is indecomposable). Also, under the assumption, there is a one-to-one correspondence between W and the set W' of elements of the Grothendieck group G of $\mathscr{C}$ given by $\operatorname{ch} M(x \cdot \lambda)$, $x \in W$, so we can identify them. Since W acts on Λ, hence on G, by assumption, we see that W fixes the subgroup $\mathbf{Z}[W]$ of G. Since each F_x^K $(x \in W)$ commutes with the W action, F_x^K can be viewed as an operator of right multiplication on $\mathbf{Z}[W]$ by

$$f_x = F_x^K(\operatorname{ch} M(\lambda)) = F_x^K(e) \in \mathbf{Z}[W].$$

With this convention, if

$$f_x = \sum_{v \in W} m_v v, \qquad m_v \in \mathbf{Z}_+,$$

then by the duality condition $(P(x \cdot \lambda) : M(v \cdot \lambda)) = m_v$. The elements of $\mathbf{Z}[W]$ can be viewed as functions on W as follows: if

$$g = \sum_{x \in W} a_x x,$$

then for $y \in W$,

$$g(y) = \sum_{x \in W} a_x \delta_{x,y},$$

where $\delta_{x,y} = 1$ if $x = y$, and $\delta_{x,y} = 0$ otherwise.

Therefore since the duality condition holds, for any $x \in W$, we have

$$f_x(y) = (P(y \cdot \lambda) : M(x \cdot \lambda)) = [M(x \cdot \lambda) : L(y \cdot \lambda)].$$

In particular, we have $f_{s_j} = s_j + e$ for any $s_j \in S$. Let

$$G_y = F_{s_1} \circ F_{s_2} \circ \cdots \circ F_{s_k}.$$

Then G_y is a BG-functor of $\mathscr{C}$ and satisfies (i)–(iii). In fact,

$$G_y^K(e) = (s_1 + e) \cdot (s_2 + e) \cdots (s_k + e),$$

and any element that appears on the right-hand side is of the form $x_1 x_2 \cdots x_k$, with $x_j = s_j$ or e. Comparing this fact with the definition of $\mathscr{S}_y$, we see that (iii) holds. (ii) follows from the fact that any $x \in W$ such that $e \leq x \leq y = s_1 \cdots s_k$ is of the form $s_{j_1} \cdots s_{j_t}$, where $(j_1, \ldots, j_t)$ is a subsequence of $(1, \ldots, k)$ (see [H]). Q.E.D.

Under the assumption of Proposition 1, each $y \in W$ corresponds to a unique element f_y of $\mathbf{Z}[W]$, these elements can be viewed both as operators of $\mathbf{Z}[W]$ and as functions on $\mathbf{Z}[W]$. They have the following properties (see [BG 4.5]):

(i) $f_y(x) \in \mathbf{Z}_+$ and $f_y(x) \geq f_y(x')$ for $x' \geq x$ and $f_y(x) > 0$ iff $y \geq x$ and $f_y(y) = 1$.

(ii) If $s \in S$ is a reflection, then $f_s(s) = f_s(e) = 1$, and $f_s(x) = 0$ if $x \neq s$, e.

(iii) If $y > x$ and $\ell(y) = \ell(x) + 1$, then $f_y(x) = 1$.

(iv) Let $s \in S$, $y \in W$ be such that $sy < y$. Then $f_y(sx) = f_y(x)$ for any $x \in W$. Similarly, if $ys < y$, then $f_y(xs) = f_y(x)$.

(v) $f_y(x) = f_{y^{-1}}(x^{-1})$ for any $y, x \in W$.

From the proof of Proposition 1, we see that for a fixed reduced expression of y, there is a one-to-one correspondence between the terms of $G_y^K(e)$ and $\mathscr{S}_y$, so we can identify them. Also note that since the composition of finite number of BG-functors is again a BG-functor, G_y has decomposition

$$G_y = F_y \oplus_{x \in E} m_{y,x} F_x, \tag{2.1.1}$$

for a subset E of W consisting of some $x \in W$ such that $x < y$ (E may be empty) and some $m_{y,x} \in \mathbf{Z}_+$. Therefore,

$$G_y^K(e) = f_y + \sum_{x \in E} m_{y,x} f_x. \tag{2.1.2}$$

Proposition 2. *Notation is as above. Assume further that the Kazhdan-Lusztig polynomials associated to W have nonegative coefficients. Suppose that for any $y \in W$ and a chosen reduced expression $y = s_1 \cdots s_k$, the following condition holds: $\mathbf{x} = (x_1, \ldots, x_k)(\in \mathscr{S}_y)$ does not appear in f_y (i.e., $\pi(\mathbf{x})$ does not contribute to a term for f_y) $\Leftrightarrow$ there is $\mathbf{z} \in \mathscr{S}_y$ with $d(\mathbf{z}) > (\ell(y) - \ell(\mathbf{x}) - 1)/2$ such that* (i) *$z = \pi(\mathbf{z}) \in E$, and* (ii) *$\pi(\mathbf{x})$ appears in f_z. Then the Kazhdan-Lusztig conjecture is true.*

Proof. Since there is a one-to-one correspondence between $\mathscr{S}_y$ and the terms in

$$G_y^K(e) = f_y + \sum_{x \in E} m_{y,x} f_x,$$

we can define $\delta(y)$ to be the subset of $\mathscr{S}_y$ corresponding to f_y. It is easy to see that these sets $\delta(y)$, $y \in W$, satisfy (1) and (2) of (1.4). On the other hand, since the Kazhdan-Lusztig polynomials have nonnegative coefficients, Algorithm 4.11 in [D] applies. Thus one can find a minimal subset $\mathscr{E}_{\min}$ of $\mathscr{S}_y$ which defines $P_{x,y}$ [D, Theorem 4.12]. By comparing the algorithm in [D] and our definition of $\delta(y)$, we see that we can choose $\mathscr{E}_{\min} = \delta(y)$. Hence the Kazhdan-Lusztig conjecture is true. Q.E.D.

3. Some equivalent conditions to the Kazhdan-Lusztig conjecture

Let $\mathscr{C}$ be a highest weight category with a Coxeter group $\langle W, S\rangle$ acting on the poset Λ satisfying conditions (1)–(3) in section 2.

Let $\mathscr{H}$ be the Hecke algebra corresponding to W. Recall that the Kazhdan-Lusztig conjecture says

$$[M(x\cdot\lambda) : L(y\cdot\lambda)] = P_{x,y}(1).$$

Theorem. *Under the assumptions of Proposition* 2.1, *the following are equivalent:*

(i) *The Kazhdan-Lusztig conjecture is true.*

(ii) *There is a homomorphism* $\psi : \mathscr{H} \to \mathbf{Z}[W]$ *such that* $\psi(q) = 1$ *and* $\psi(C'_x) = f_x$ *, for any* $x \in W$.

(iii) *The nonzero coefficients of all Kazhdan-Lusztig polynomials given by* $\mathscr{H}$ *are positive integers, and for any* $y \in W$ *and a fixed reduced expression* $y = s_1 s_2 \cdots s_k$, *the subset* E *of* W *defined by* (2.1.1) *equals the subset of* B *of* W *defined by* (1.4.2) *and* $m_x(\delta(y)) = m_{y,x}$.

Proof. (i) $\Rightarrow$ (ii). Assume that the Kazhdan-Lusztig conjecture is true. Then by the duality condition, we have

$$(P(y\cdot\lambda) : M(x\cdot\lambda)) = [M(x\cdot\lambda) : L(y\cdot\lambda)] = P_{x,y}(1).$$

Let ϕ be the isomorphism $\mathscr{H}/(q-1) \cong \mathbf{Z}[W]$, and let $p : \mathscr{H} \to \mathscr{H}/(q-1)$ be the canonical homomorphism. Then $\psi = \phi \circ p$ is the homomorphism we are looking for, since

$$\psi(C'_y) = \sum_{x \leq y} P_{x,y}(1)x = f_y.$$

(ii) $\Rightarrow$ (iii). Assume that $\psi : \mathscr{H} \to \mathbf{Z}[W]$ is a homomorphism such that $\psi(q) = 1$, $\psi(C'_x) = f_x$ for all $x \in W$. Then since C'_x , $x \in W$ form a basis of $\mathscr{H}$, f_x, $x \in W$ form a basis of $\mathbf{Z}[W]$ over $\mathbf{Z}$. If $s \in S$, then we have

$$\psi(C'_x \cdot C'_s) = f_x \cdot f_s,$$

where $C'_s = q^{-1/2}(T_s + T_e)$. Now if $\ell(xs) > \ell(x)$, then setting $xs = y$, we have

$$f_x \cdot f_s = f_y + \sum_{z<y} a_z f_z \quad \text{for some } a_z \in \mathbf{Z}_+.$$

On the other hand we have (see [KL, section 2])

$$C'_x \cdot C'_s = C'_y + \sum_{z<y} \mu(z, y) C'_z, \tag{3.1}$$

Thus by applying ψ to (3.1), we have

$$f_x \cdot f_s = f_y + \sum_{z<y} \mu(z, y) f_z.$$

(We adopt the convention that $\mu(z, y)$ may be zero.) Hence $a_z = \mu(z, y)$. This implies in particular that the $\mu(z, y)$'s are all nonnegative integers. Therefore by [D, Corollary 3.8], the polynomials L_z introduced by [D, Proposition 3.7] have nonnegative integral coefficients for all $z \in W(y) = \{z \in W : z \leq y\}$. Hence by [D, Theorem 4.12], the D-condition is satisfied and we can find a D-set $\delta(y)$ of $\mathscr{S}_y$ such that

$$P_{x,y} = \sum_{\substack{\mathbf{x} \in \delta(y) \\ \pi(\mathbf{x}) = x}} q^{d(\mathbf{x})}. \tag{3.2}$$

This implies in particular that all $P_{x,y}$ have nonnegative integer coefficients. Also, since (3.2) holds, from

$$\pi(\mathscr{S}_y) = \pi(\delta(y)) + \pi\left(\bigcup_{i=1}^{t} \iota_i(\mathscr{B}_i)\right),$$

we have

$$G_y^K(e) = f_y + \sum_{x \in B} m_x(\delta(y)) f_x.$$

Since $\{f_x, x \in W\}$ is a basis of $\mathbf{Z}[W]$, by comparing the above identity with (2.1.2), we see that $E = B$ and $m_x(\delta(y)) = m_{y,x}$ as desired.

(iii) $\Rightarrow$ (i). If all coefficients of $P_{x,y}$ are nonnegative, then by [D, Theorem 4.12], one can find $\delta(y)$ such that (3.2) holds. On the other hand, $E = B$ and $m_x(\delta(y)) = m_{y,x}$ imply that there is a one-to-one correspondence between the elements of $\delta(y)$ and the terms of f_y. So for $x \leq y$,

$$\begin{aligned} P_{x,y}(1) &= |\{\mathbf{x} \in \delta(y) : \pi(\mathbf{x}) = x\}| \\ &= f_y(x) \\ &= (P(y \cdot \lambda) : M(x \cdot \lambda)) \\ &= [M(y \cdot \lambda) : L(x \cdot \lambda)]. \end{aligned}$$

That is, the Kazhdan-Lusztig conjecture is true. Q.E.D.

Acknowledgment

The author wishes to express his indebtedness to V. V. Deodhar. The author learned these materials under his guidance when the author was a Ph.D. student at Indiana University.

References

[BB] A. Beilinson and I. Bernstein, *Localization de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), 15–18.

[BG] J. Bernstein and S. I. Gelfand, *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras*, Compositio Math. **41** (1980), 245–285.

[BGG] J. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Category of $\mathfrak{g}$-modules*, Functional Anal. Appl. **10** (1976), 87–92.

[BK] J. L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), 387–410.

[CPS1] E. Cline, B. Parshall, and L. Scott, *Finite dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 85–99.

[CPS2] ——, *Duality in highest weight categories*, Contemp. Math., vol. 82, Amer. Math. Soc., Providence, RI, 1989, pp. 7–22.

[D] V. V. Deodhar, *A combinatorial setting for questions in Kazhdan-Lusztig theory*, Geom. Dedicata **36** (1990), 95–119.

[DS] J. Du and L. Scott, *Lusztig conjectures, old and new,* I, preprint.

[H] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Univ. Press, London, 1990.

[KL1] D. Kazhdan and G. Lusztig, *Representation of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[KL2] ——, *Schubert varieties and Poincare duality*, Proc. Sympos. Pure Math., vol. 36, Amer. Math. Soc., Providence, RI, 1980, pp. 185–203.

Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201

E-mail address: ymzou@convex.csd.uwm.edu