

## THE OPERATOR $(\operatorname{sgn} x) \frac{d^2}{dx^2}$ IS SIMILAR TO A SELFADJOINT OPERATOR IN $L^2(\mathbb{R})$

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**ABSTRACT.** Krein space operator-theoretic methods are used to prove that the operator  $(\operatorname{sgn} x) \frac{d^2}{dx^2}$  is similar to a selfadjoint operator in the Hilbert space  $L^2(\mathbb{R})$ .

Let  $L$  be a symmetric ordinary differential expression. Spectral properties of the operators associated with the weighted eigenvalue problem  $Lu = \lambda wu$  have been studied extensively. When  $w$  is positive, this problem leads to a selfadjoint problem in the Hilbert space  $L^2(w)$ . In recent years there has been considerable interest in the case when  $w$  changes sign; for a survey see [5] and also [2]. In this case the problem may have nonreal and nonsemisimple spectrum. Since the problem is symmetric with respect to an indefinite scalar product, it is natural to consider the problem in the associated Krein space. The corresponding operator can be studied using the spectral theory of definitizable selfadjoint operators in Krein spaces. For definitions and basic results of this theory see [4].

Let  $(\mathcal{K}, [\cdot|\cdot])$  be a Krein space,  $A$  a definitizable operator in  $\mathcal{K}$ , and  $E$  the spectral function of  $A$ . Of particular interest are the so-called critical points of  $A$  where its spectral properties are different from the spectral properties of a selfadjoint operator in Hilbert space. Definitizable operators may have at most finitely many critical points. Significantly different behavior of the spectral function occurs at singular critical points in any neighborhood of which the spectral function is unbounded. The critical points which are not singular are called regular. The simplest class of definitizable operators are positive operators with nonempty resolvent set: an operator  $A$  is positive if  $[Ax, x] \geq 0$ ,  $x \in \mathcal{D}(A)$ . The spectrum of a positive operator is real; only 0 may be a non-semisimple eigenvalue; only 0 and  $\infty$  may be critical points. Moreover, if 0 and  $\infty$  are not singular critical points and if 0 is not an eigenvalue, then the operator  $A$  is a selfadjoint operator in the Hilbert space  $(\mathcal{K}, [(E(\mathbb{R}_+) - E(\mathbb{R}_-)) \cdot |\cdot|])$ ; see [4, Theorem 5.7].

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In this note we consider a model problem where  $Ly = -y''$  and  $w(x) = \operatorname{sgn} x$ . Even in this simple case it does not appear to be easy to obtain results by classical methods. Our method extends to more general  $L$  and  $w$ .

Let  $\mathcal{K} = L^2(\mathbf{R})$  be the Krein space with the scalar product  $[f|g] = \int_{\mathbf{R}} f(x)g(x)\operatorname{sgn} x dx$ . The multiplication operator  $Jy = (\operatorname{sgn} x)y$  is a fundamental symmetry on  $\mathcal{K}$ . Let  $Ay = -(\operatorname{sgn} x)y''$  for  $y \in \mathcal{D}(A) = H^2(\mathbf{R})$ .

**Proposition 1.** *The operator  $A$  is a positive definitizable operator in  $\mathcal{K}$ .*

*Proof.* We only have to prove that the resolvent set of  $A$  is nonempty. Since  $B = JA$  is a closed operator, the operator  $A$  is closed. Therefore, it is sufficient to prove that  $B - iJ$  is a bijection of  $H^2(\mathbf{R})$  onto  $L^2(\mathbf{R})$ . Let  $g \in L^2(\mathbf{R})$ . The boundary value problems

$$-y''(x) \pm iy(x) = g(x), \quad x \in \mathbf{R}_{\mp}, \quad y(0) = 0, \quad y \in L^2(\mathbf{R}_{\mp}),$$

have unique solutions  $y_{\mp}$  in  $H^2(\mathbf{R}_{\mp})$ . Let  $\psi_{\mp}(x) = \exp(\frac{x}{\sqrt{2}}(\pm 1 + i))$ . Then the general solution of the equation

$$-y''(x) \pm iy(x) = g(x), \quad x \in \mathbf{R}_{\mp}, \quad y \in L^2(\mathbf{R}_{\mp}),$$

is given by  $y_{\mp} + c_{\mp}\psi_{\mp}$ . Therefore, every solution  $z \in H^2(\mathbf{R})$  of the equation

$$-z'' - i(\operatorname{sgn} x)z = g$$

must satisfy

$$z(x) = \begin{cases} y_-(x) + c_- \psi_-(x), & x \in \mathbf{R}_-, \\ y_+(x) + c_+ \psi_+(x), & x \in \mathbf{R}_+, \end{cases}$$

for some complex numbers  $c_-$  and  $c_+$ . The continuity of  $z$  and  $z'$  at 0 yields

$$c_- = c_+ = \frac{y'_+(0) - y'_-(0)}{\sqrt{2}}.$$

Consequently,  $B - iJ$  and, therefore, also  $A - iI$  are bijections of  $H^2(\mathbf{R})$  onto  $L^2(\mathbf{R})$ . Thus,  $i \in \rho(A)$ .  $\square$

**Proposition 2.** *The operator  $A$  has no eigenvalues. Its spectrum coincides with  $\mathbf{R}$ . The only critical points of  $A$  are 0 and  $\infty$ .*

*Proof.* Denote by  $U_{\alpha}$ ,  $\alpha \in \mathbf{R} \setminus \{0\}$ , the dilation operator:  $(U_{\alpha}f)(x) = f(\alpha x)$ . Then  $U_{\alpha}$  is a bounded operator with the bounded inverse  $U_{1/\alpha}$ . We have  $U_{\alpha}^{-1}AU_{\alpha} = (\operatorname{sgn} \alpha)\alpha^2 A$ . This implies that the spectrum of  $A$  is invariant under multiplication by real nonzero numbers. Since  $A$  is unbounded, its spectrum is unbounded; see [4, Corollary 2 of Lemma 2.2]. Therefore,  $\mathbf{R} \setminus \{0\} \subset \sigma(A)$ . Because  $0 \in \sigma(B)$ , it follows that  $0 \in \sigma(A)$ .

Since  $p(\lambda) = \lambda$  is a definitizing polynomial of  $A$ , Theorem 3.1(4) and Proposition 4.2 in [4] imply that 0 and  $\infty$  are the only critical points of  $A$ .  $\square$

Let  $\mathcal{D} = \mathcal{D}(B) = H^2(\mathbf{R})$  and  $\mathcal{R} = \mathcal{R}(B) = \{y'' : y \in H^2(\mathbf{R})\}$ . Let  $\mathcal{D}_0$  be the space of all functions  $y \in \mathcal{D}$  such that  $y(0) = y'(0) = 0$ , and put  $\mathcal{R}_0 = B\mathcal{D}_0$ . The following lemma is evident.

**Lemma 3.** *The spaces  $\mathcal{D}_0$  and  $\mathcal{R}_0$  are invariant under the fundamental symmetry  $J$ .*

**Lemma 4.** *There exist positive bounded and boundedly invertible operators  $X$  and  $T$  in the Hilbert space  $L^2(\mathbf{R})$  such that  $X\mathcal{R} \subset \mathcal{R}_0$  and  $T\mathcal{D} \subset \mathcal{D}_0$ .*

*Proof.* We construct only the operator  $X$ . The construction of  $T$  is similar; see [2, Lemma 3.2]. Let  $\alpha_j, j = 1, 2, 3, 4$ , be distinct positive real numbers. Let  $\xi_j, j = 1, 2, 3, 4$ , be the solution of the linear system

$$\begin{aligned} \xi_1 + \xi_2 + \xi_3 + \xi_4 &= 1, \\ \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3 + \alpha_4 \xi_4 &= 1, \\ \alpha_1^2 \xi_1 + \alpha_2^2 \xi_2 + \alpha_3^2 \xi_3 + \alpha_4^2 \xi_4 &= -1, \\ \alpha_1^3 \xi_1 + \alpha_2^3 \xi_2 + \alpha_3^3 \xi_3 + \alpha_4^3 \xi_4 &= -1. \end{aligned}$$

Let

$$Z = \sum_{j=1}^4 \xi_j U_{\alpha_j}, \quad Z_1 = \sum_{j=1}^4 \xi_j \alpha_j^3 U_{1/\alpha_j}, \quad Y = \sum_{j=1}^4 \xi_j \alpha_j^2 U_{\alpha_j}.$$

Since  $U_\alpha^* = (1/\alpha)U_{1/\alpha}$ , we find that

$$Y^* = \sum_{j=1}^4 \xi_j \alpha_j U_{1/\alpha_j}.$$

Here  $*$  denotes adjoint in the Hilbert space  $L^2(\mathbf{R})$ . As linear combinations of bounded operators, the operators  $Z, Z_1, Y$ , and  $Y^*$  are bounded in  $L^2(\mathbf{R})$ . Note that  $YB = BZ$  and  $Y^*B = BZ_1$  on  $\mathcal{D}$ . Therefore,

$$(Y^*Y + I)B = B(Z_1Z + I) \quad \text{on } \mathcal{D}.$$

From the choice of  $\xi_j, j = 1, 2, 3, 4$ , it follows that

$$(Zu)^{(k)}(0) = u^{(k)}(0) \quad \text{and} \quad (Z_1u)^{(k)}(0) = -u^{(k)}(0), \quad k = 0, 1,$$

for all  $u \in \mathcal{D}$ . Therefore,  $(Z_1Z + I)\mathcal{D} \subset \mathcal{D}_0$ . Put  $X = Y^*Y + I$ . Then

$$X\mathcal{R} = XB\mathcal{D} = B(Z_1Z + I)\mathcal{D} \subset B\mathcal{D}_0 = \mathcal{R}_0.$$

Therefore,  $X$  has all the required properties.  $\square$

**Corollary 5.** *There exist positive bounded and boundedly invertible operators  $W$  and  $V$  in the Krein space  $K$  such that  $W\mathcal{D} \subset \mathcal{D}$  and  $V\mathcal{R} \subset \mathcal{R}$ .*

*Proof.* Put  $W = JT$  and  $V = JX$ , and apply Lemmas 3 and 4.  $\square$

**Theorem 6.** *The points 0 and  $\infty$  are regular critical points of  $A$ .*

*Proof.* The regularity of the critical point  $\infty$  follows from Corollary 5 and [3, Theorem 2.1, (iii)  $\Rightarrow$  (i)]; see also [1, Theorem 3.9].

The operator  $A^{-1}$  is also a positive definitizable operator. In order to prove that 0 is regular critical point of  $A$  it is sufficient to show that  $\infty$  is a regular critical point of  $A^{-1}$ . The operator  $JA^{-1}J$  is a positive definitizable operator, similar to  $A^{-1}$ . Corollary 5 implies that  $V\mathcal{R}(B) \subset \mathcal{R}(B)$ , which is equivalent to  $V\mathcal{D}(JA^{-1}J) \subset \mathcal{D}(JA^{-1}J)$ . By Corollary 5 and [3, Theorem 2.1 (iii)  $\Rightarrow$  (i)] it follows that  $\infty$  is a regular critical point of  $JA^{-1}J$ .  $\square$

**Corollary 7.** *The operator  $A$  is similar to a selfadjoint operator in the Hilbert space  $L^2(\mathbf{R})$ .*

*Proof.* As we mentioned at the beginning of this note, since 0 and  $\infty$  are not singular critical points of the operator  $A$ , it follows from [4, Theorem 5.7] that  $A$  is selfadjoint in the Hilbert space  $(\mathcal{H}, [(E(\mathbf{R}_+) - E(\mathbf{R}_-)) \cdot |\cdot|])$ . Here

$\mathcal{H} = L^2(\mathbf{R})$ . Since the operator  $J(E(\mathbf{R}_+) - E(\mathbf{R}_-))$  is bounded and boundedly invertible, the Hilbert spaces  $L^2(\mathbf{R})$  and  $(\mathcal{H}, [(E(\mathbf{R}_+) - E(\mathbf{R}_-)) \cdot | \cdot])$  have equivalent norms. Therefore, the operator  $A$  is similar to a selfadjoint operator in  $L^2(\mathbf{R})$ .  $\square$

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