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NUMERICAL RADIUS PERSERVING OPERATORS ON $B(H)$

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ABSTRACT. Let H be a Hilbert space over $\mathbb{C}$ and let $B(H)$ denote the vector space of all bounded linear operators on H. We prove that a linear isomorphism $T : B(H) \to B(H)$ is numerical radius-preserving if and only if it is a multiply of a C^*-isomorphism by a scalar of modulus one.

1. INTRODUCTION

Let H be a Hilbert space over $\mathbb{C}$ and let $B(H)$ denote the vector space of all bounded linear operators on H. For every A in $B(H)$, the numerical range and the numerical radius of T are defined respectively by

$$W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\} ,$$
$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\} .$$

It is well known that $w(\cdot)$ is a norm on $B(H)$ and that this norm is equivalent to the usual operator norm. (See [4, p. 117].) A classical theorem of Kadison [4, Theorem 7] asserts that every linear isomorphism on $B(H)$ which is isometric with respect to the operator norm is a C^*-isomorphism followed by left multiplication by a fixed unitary operator. A C^*-isomorphism is a linear isomorphism of $B(H)$ such that $T(A^*) = T(A)^*$ for all A in $B(H)$ and $T(A^n) = T(A)^n$ for all selfadjoint A in $B(H)$ and all natural number n. A description of C^*-isomorphisms on $B(H)$ can be obtained. First of all we have from [6, Corollary 11] that a C^*-isomorphism on $B(H)$ is either a *-isomorphism or a *-anti-isomorphism. Suppose that T is an algebra isomorphism on $B(H)$. Then by [3, Theorem 2], there is an invertible operator V on H such that $T(A) = VAV^{-1}$ for all A in $B(H)$. If we also assume that $T(A^*) = T(A)^*$ for all A in $B(H)$, then $VA^*V^{-1} = (V^{-1})^*A^*V^*$ and hence $(V^*V)A^* = A^*(V^*V)$ for all A in $B(H)$. It follows that V^*V is a scalar multiple of the identity operator I. Say $V^*V = kI$. As V^*V is always a positive operator and k cannot be zero, $k > 0$. Let $U = \frac{1}{\sqrt{k}}V$. Then U is unitary and $T(A) = UAU^*$ for all A in $B(H)$. For a *-anti-isomorphism T, it can be shown (e.g., see [5, Remark 2]) that there is a unitary operator U in $B(H)$ such that $T(A) = UA^tU^*$ for all A in $B(H)$, where A^t denotes the transpose

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of A relative to a fixed orthonormal basis of H. Clearly operators of these two types are C^*-isomorphisms.

Let us turn to numerical range and numerical radius. Pellegrini [9, Theorem 3.1] proved that an operator T on $B(H)$ is a C^*-isomorphism exactly when T preserves the "numerical range" of each element in $B(H)$. It should be noted that Pellegrini obtained his result in a general Banach algebra, and his definition of numerical range is different from ours. In fact, for each A in $B(H)$, the "numerical range" of A defined by Pellegrini reduces to the closure of $W(A)$. When the underlying space H is finite-dimensional, $W(A)$ is compact and hence the two sets are identical. Despite the discrepancy we still have that T is a C^*-isomorphism if and only if $W(T(A)) = W(A)$ for every A in $B(H)$. For simplicity we shall call an operator T with the latter property numerical range-preserving. Likewise we say that T is numerical radius-preserving if $w(T(A)) = w(A)$ for all A in $B(H)$.

In the finite-dimensional situation, the above result was extended by Li. In [1, Theorem 1] he proved that T is numerical radius-preserving if and only if T is a scalar multiple of a C^*-isomorphism by a complex number of modulus one. It is immediate that if T is numerical range-preserving, then T is numerical radius-preserving and hence the scalar in question is one. In this note we prove that the conclusion of Li remains valid without the dimension constraint.

2. Results

In what follows T denotes a linear isomorphism on $B(H)$ which is numerical radius-preserving on $B(H)$. We shall prove that T maps the identity mapping I to a scalar multiple of I. The scalar is necessarily of modulus one. Multiplying by the complex conjugate of the scalar, we get a numerical radius-preserving operator T_1 with an additional property that $T_1(I) = I$. The result is concluded by showing that T_1 is a C^*-isomorphism.

We begin with a lemma which describes scalar multiples of I in terms of numerical radius. Let $\Lambda = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Lemma 1. *An operator $A \in B(H)$ is a scalar multiple of I if and only if for every $B \in B(H)$, there is a $\lambda \in \Lambda$ such that $w(A + \lambda B) = w(A) + w(B)$.*

Proof. It is clear that if A is a scalar multiple of I, then A satisfies the condition. For the converse we borrow the idea from Li and Tsing [2, p. 40]. We first show that elements in $W(A)$ are of constant modulus; it follows then from the convexity of $W(A)$ ([4, p. 113]) that the set is a singleton. Hence A is a scalar multiple of the identity I. Now assume that there is an x in H, $\|x\| = 1$, and $|\langle Ax, x\rangle| < w(A)$. Let B be the orthogonal projection onto the linear span of x. Then $w(B) = 1$. Fix any r such that $|\langle Ax, x\rangle| < r < w(A)$. We can find an $\varepsilon > 0$ such that $|\langle Ay, y\rangle| < r$ whenever $\|y - x\| < \varepsilon$. In fact $|\langle Ay, y\rangle| < r$ if there is a $\lambda \in \Lambda$ such that $\|y - \lambda x\| < \varepsilon$. Suppose that $y \in H$, $\|y\| = 1$, and $\|y - \lambda x\| \geq \varepsilon$ for every $\lambda \in \Lambda$. Then

$$\varepsilon^2 \leq \langle y - \lambda x, y - \lambda x\rangle = 2 - 2\mathrm{Re}\langle y, \lambda x\rangle \quad \text{for every} \quad \lambda \in \Lambda .$$

It follows that $\langle y, x\rangle \leq 1 - \frac{1}{2}\varepsilon^2$. Let $k = \min\{r + 1, w(A) + 1 - \frac{1}{2}\varepsilon^2\}$. Then for every $\lambda \in \Lambda$ and $y \in H$ with $\|y\| = 1$, we have

$$\left|\langle (A + \lambda B)y, y\right| \leq \left|\langle Ay, y\rangle\right| + \left|\langle y, x\rangle\right| \leq k .$$

Hence $w(A + \lambda B) < w(A) + w(B)$. $\square$

By the above lemma $T(I) = \lambda I$. Clearly we have $\lambda \in \Lambda$. Let $T_1 = \bar{\lambda} T$. Then $T_1(I) = I$. We need the following definitions. By a state on $B(H)$ we mean as usual a bounded linear functional ρ on $B(H)$ such that $\rho(I) = \|\rho\| = 1$. The set S of all states is called the state space of $B(H)$. A bounded linear operator $T : B(H) \to B(H)$ is said to be state-preserving if its adjoint T' satisfies $T'(S) \subseteq S$. By [9, Theorem 2.3 and Theorem 3.1], T is a C^*-isomorphism if and only if it is state-preserving. Let x be a unit vector in H. The linear functional ρ_x given by

$$\rho_x(A) = \langle Ax, x \rangle \quad \text{for every} \quad A \in B(H)$$

is a state of $B(H)$. States of this form are called vector states.

Lemma 2. *The operator* T_1 *is state-preserving.*

Proof. Let w' denote the norm in $B(H)'$ dual to the numerical radius. Then $w'(\rho) \geq \|\rho\|$ for every ρ in $B(H)'$. As T_1 is numerical radius-preserving, $w'(T_1'(\rho)) = w'(\rho)$ for every ρ in $B(H)'$. If ρ_x is a vector state, then $w'(\rho_x) = 1$ and hence $\|T_1'(\rho_x)\| \leq w'(T_1'(\rho_x)) = 1$. But $T_1'(\rho_x)(I) = \rho_x(T_1(I)) = \rho_x(I) = 1$. It follows that $T_1'(\rho_x)$ is a state of $B(H)$. By [4, Corollary 4.3.10] the state space is the closed convex hull of the vector states in the weak*-topology. This together with the fact that T_1' is continuous in the weak*-topology entail that T_1 is state-preserving. □

By Lemma 1 and Lemma 2, we have proved

Theorem. *A linear isomorphism* T *on* $B(H)$ *is numerical radius-preserving if and only if* T *is a multiple of a* C^**-isomorphism by a scalar of modulus one.*

In [1] Li also studied a numerical radius-preserving real-linear operator on the selfadjoint elements in $B(H)$. He proved ([1, Theorem 2]) that such an operator is the restriction of a C^*-isomorphism on $B(H)$ multiplied by ± 1. Let us remark that as the numerical radius and the operator norm coincide on selfadjoint operators, this result can alternatively be deduced from [7, Theorem 2].

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