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THE MINIMAL NORMAL μ-COMPLETE FILTER ON $P_\kappa\lambda$

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ABSTRACT. We introduce the closed (resp. strongly closed) μ-unbounded filter $CF^\mu_{\kappa\lambda}$ (resp. $SCF^\mu_{\kappa\lambda}$) for a regular cardinal $\mu \leq \kappa$, which is a μ-complete analog of the closed (resp. strongly closed) unbounded filter on $P_\kappa\lambda$. We give their simple characterizations for the case $\mu = \omega_1$. We also study generators of $CF^\nu_{\kappa\lambda}$ (resp. $SCF^\nu_{\kappa\lambda}$) over $CF^\mu_{\kappa\lambda}$ (resp. $SCF^\mu_{\kappa\lambda}$) for the case $\omega < \mu < \nu$.

0. INTRODUCTION

The closed unbounded filter on $P_\kappa\lambda$, denoted by $CF_{\kappa\lambda}$, was introduced by Jech [5] as a natural generalization of that on a regular cardinal κ. Its fundamental characterization as the minimal normal κ-complete filter was established by Carr [1]. At the same time, she reformulated a result of Menas [8] which indicates the complexty of $P_\kappa\lambda$ relative to κ, as the difference between $CF_{\kappa\lambda}$ and the strongly closed unbounded filter $SCF_{\kappa\lambda}$ introduced by herself. This simple fact motivates our work in the present paper and [10].

In this note, we introduce their μ-complete analogs for a regular cardinal $\mu \leq \kappa$, i.e., the closed μ-unbounded filter $CF^\mu_{\kappa\lambda}$ and the strongly closed μ-unbounded filter $SCF^\mu_{\kappa\lambda}$. Our main concern is to stress the difference between $CF^\mu_{\kappa\lambda}$ and $SCF^\mu_{\kappa\lambda}$ through investigating the forms of their generators.

First we consider the case $\mu = \omega_1$. For their own purposes, several people [2, 4, 6, 9] have already considered similar problems for $CF_{\kappa\lambda}$, or more specifically for $CF_{\omega_1\lambda}$. Among them, Matet [7] gave the simplest solutions to both cases. By a rather different argument from his, we show that $CF^{\omega_1}_{\kappa\lambda}$ is generated by the sets of the form $\{x : f''x^2 \subset x\}$, where $f : \lambda^2 \to \lambda$. This includes both his results as immediate corollaries. We also show that $SCF^{\omega_1}_{\kappa\lambda}$ can be characterized by two unary functions on λ but not by one unary function, in contrast to $CF^{\omega_1}_{\kappa\lambda}$.

The same problem as above is meaningless for the case $\mu > \omega_1$. Instead we study generators of $CF^\nu_{\kappa\lambda}$ (resp. $SCF^\nu_{\kappa\lambda}$) over $CF^\mu_{\kappa\lambda}$ (resp. $SCF^\mu_{\kappa\lambda}$) for

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TABLE 1

$$
\begin{array}{ccccccccc}
CF^{\mu^+} & & & & & & & & \\
\| & & & & & & & & \\
CF^{\mu}+A & = & CF^{\mu}+B & = & CF^{\mu}+C & \supset & CF^{\mu}+D & \supsetneq & CF^{\mu} \\
\| & & \| & & \| & & \cup & & \not\cup \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\| & & \| & & \| & & \cup & & \not\cup \\
CF^{\omega_1}+A & = & CF^{\omega_1}+B & = & CF^{\omega_1}+C & \supset & CF^{\omega_1}+D & \supsetneq & CF^{\omega_1} \\
\| & & \| & & \| & & \| & & \| \\
CF^{\omega}+A & = & CF^{\omega}+B & = & CF^{\omega}+C & \supset & CF^{\omega}+D & \supsetneq & CF^{\omega}
\end{array}
$$

$$
\begin{array}{ccccccccc}
SCF^{\mu^+} & & & & & & & & \\
\| & & & & & & & & \\
SCF^{\mu}+A & \supsetneq & SCF^{\mu}+B & \supsetneq & SCF^{\mu}+C & \supsetneq & SCF^{\mu}+D & \supsetneq & SCF^{\mu} \\
\| & & \not\cup & & \not\cup & & \not\cup & & \not\cup \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\| & & \not\cup & & \not\cup & & \not\cup & & \not\cup \\
SCF^{\omega_1}+A & \supsetneq & SCF^{\omega_1}+B & \supsetneq & SCF^{\omega_1}+C & \supsetneq & SCF^{\omega_1}+D & \supsetneq & SCF^{\omega_1} \\
\| & & \| & & \| & & \| & & \| \\
SCF^{\omega}+A & \supsetneq & SCF^{\omega}+B & \supsetneq & SCF^{\omega}+C & \supsetneq & SCF^{\omega}+D & \supsetneq & SCF^{\omega}
\end{array}
$$

$\omega < \mu < \nu$. For example, it has been known to people including authors of [2, 4] that the set $\{x : x \cap \kappa \in \kappa\}$ generates $CF_{\kappa\lambda}$ over $CF_{\kappa\lambda}^{\omega_1}$. We investigate whether this statement would hold for other simply definable subsets of $P_\kappa\lambda$.

Table 1 summarizes our result. Subscripts κ and λ are dropped for notational simplicity. We also abbreviate by A, B, C, and D the sets $\{x : \sigma_\mu'' x \subset Px\}$, $\{x : x \cap \mu^+ \in \mu^+\}$, $\{x : \mu \subset x\}$, and $\{x : |x| \geq \mu\}$ respectively, where $\omega_1 < \mu < \kappa$.

1. PRELIMINARIES

Fix a regular cardinal $\kappa > \omega$ and a cardinal $\lambda > \kappa$. We begin by recalling some basic notions on $P_\kappa\lambda = \{x \subset \lambda : |x| < \kappa\}$. $F \subsetneq PP_\kappa\lambda$ is called a filter if it is closed under intersections and supersets and is fine (i.e., $\{x : \alpha \in x\} \in F$ for any $\alpha < \lambda$). A subset of $P_\kappa\lambda$ is said to be closed (resp. strongly closed) if it is closed under unions of chains (resp. subsets) of cardinality $< \kappa$, and is said to be unbounded if it is cofinal in the partially ordered set $(P_\kappa\lambda, \subset)$. A filter F is said to be μ-complete if it is closed under intersections of subsets of cardinality $< \mu$ and is said to be normal if it is closed under diagonal intersections (i.e., $\Delta_{\alpha<\lambda} X_\alpha = \{x : \forall\alpha \in x\ x \in X_\alpha\} \in F$ for any $\{X_\alpha : \alpha < \lambda\} \subset F$). $CF_{\kappa\lambda}$ (resp. $SCF_{\kappa\lambda}$) is the filter generated by the closed (resp. strongly closed) and unbounded subsets of $P_\kappa\lambda$.

We use μ and ν (resp. ξ and ζ) to denote an infinite regular cardinal $\leq \kappa$ (resp. an uncountable cardinal $< \kappa$). We denote by $F + X$ the filter generated by $F \cup \{X\}$, where $X \subset P_\kappa\lambda$ is positive and copositive with respect to a filter F (i.e., $P_\kappa\lambda - X$, $X \notin F$). $\alpha + \beta$ (resp. $\alpha \cdot \beta$) is the ordinal sum

(resp. product) of α and β. We fix a bijection $\pi : \lambda^2 \to \lambda$. $\sigma : \lambda \to \lambda$ and $\sigma_\xi : \lambda \to [\lambda]^\xi$ denote the successor function and the function defined by $\sigma_\xi(\alpha) = \{\alpha + \beta : \beta < \xi\}$ respectively. By $\alpha \to [\beta]^{<\omega}_{P_\gamma\delta}$ (resp. $\alpha \to [\beta]^{<\omega}_{\gamma,<\delta}$), we mean that for any $f : [\alpha]^{<\omega} \to P_\gamma\delta$ (resp. $f : [\alpha]^{<\omega} \to \gamma$) there exists $x \in [\alpha]^\beta$ with $\bigcup f''[x]^{<\omega} \neq \delta$ (resp. $|f''[x]^{<\omega}| < \delta$), where α, β, γ and δ are all cardinals.

Before introducing μ-complete analogs of $CF_{\kappa\lambda}$ and $SCF_{\kappa\lambda}$, it is appropriate to recall the following characterizations of the original versions.

1.1. **Proposition** [1, 8]. *$CF_{\kappa\lambda}$ (resp. $SCF_{\kappa\lambda}$) is generated by the sets of the form $\{x : f''x^2 \subset Px\}$ (resp. $\{x : f''x \subset Px\}$), where $f : \lambda^2 \to P_\kappa\lambda$ (resp. $f : \lambda \to P_\kappa\lambda$).*

Definition. (1) *$CF^\mu_{\kappa\lambda}$ (resp. $SCF^\mu_{\kappa\lambda}$) is the filter generated by the sets of the form $\{x : f''x^2 \subset Px\}$ (resp. $\{x : f''x \subset Px\}$), where $f : \lambda^2 \to P_\mu\lambda$ (resp. $f : \lambda \to P_\mu\lambda$).*

(2) *$X \subset P_\kappa\lambda$ is μ-unbounded in $P_\kappa\lambda$ iff $X \cap P_\mu\lambda$ is unbounded in $P_\mu\lambda$.*

The same argument as in [1, 2, 4, 8] works for the following (see [10] for a short proof of (2)).

1.2. **Proposition.** (1) *$CF^\mu_{\kappa\lambda}$ (resp. $SCF^\mu_{\kappa\lambda}$) is generated by the closed (resp. strongly closed) and μ-unbounded subsets of $P_\kappa\lambda$.*

(2) $CF^\mu_{\kappa\lambda} = SCF^\mu_{\kappa\lambda} + \{x : \pi''x^2 \subset x\}$.

(3) *$CF^\mu_{\kappa\lambda}$ is the minimal normal μ-complete filter on $P_\kappa\lambda$.*

(4) $CF^\mu_{\kappa\lambda} = CF^\omega_{\kappa\lambda} + \{x : x \cap \mu \in \mu\}$, *where* $\omega_1 < \mu$.

Proof. Let us give a combinatorial proof of (4), which is a prototype of later arguments.

Fix $f : \lambda^2 \to P_\mu\lambda$. We define $h : \lambda^2 \to P_\omega\lambda$ and show $\{x : x \cap \mu \in \mu \text{ and } h''x^2 \subset Px\} \subset \{x : f''x^2 \subset Px\}$.

First define $g : \lambda^3 \to \lambda$ by $g(\alpha, \beta, 0) = |f(\alpha, \beta)|$, $\{g(\alpha, \beta, \gamma) : 0 < \gamma \leq |f(\alpha, \beta)|\} = f(\alpha, \beta)$ and $g(\alpha, \beta, \gamma) = 0$ for $\gamma > |f(\alpha, \beta)|$. We show $\{x : x \cap \mu \in \mu \text{ and } g''x^3 \subset x\} \subset \{x : f''x^2 \subset Px\}$.

Let $x \in P_\kappa\lambda$ be closed under g and $x \cap \mu \in \mu$. Fix $\alpha, \beta \in x$. Then $0 \in x$, since $x \cap \mu \neq 0$ is an ordinal when $\alpha < \mu$ and since $0 = g(\alpha, \alpha, \alpha)$ when $\alpha \geq \mu$. Hence $|f(\alpha, \beta)| + 1 \subset x$, since $|f(\alpha, \beta)| = g(\alpha, \beta, 0) \in x \cap \mu$ is an ordinal. Thus $f(\alpha, \beta) \subset \{g(\alpha, \beta, \gamma) : \gamma \leq |f(\alpha, \beta)|\} \subset x$.

Define $h : \lambda^2 \to P_\omega\lambda$ by $h(\alpha, \beta) = \{\pi(\alpha, \beta), g(\alpha, \pi^{-1}(\beta))\}$. Then $\{x : h''x^2 \subset Px\} \subset \{x : g''x^3 \subset x\}$ by the standard argument and hence we are done. □

Several people [2, 4, 6, 9] have already characterized $CF^{\omega_1}_{\kappa\lambda}$ or $CF_{\omega_1\lambda}$ in their own ways. Let us summarize some of them.

1.3. **Proposition.** *The following are equivalent for $X \subset P_\kappa\lambda$.*

(1) $X \in CF^\omega_{\kappa\lambda}$.

(2) $X \in CF^{\omega_1}_{\kappa\lambda}$.

(3) $\{x : f''x^{<\omega} \subset x\} \subset X$ *for some* $f : \lambda^{<\omega} \to \lambda$.

(4) $\{x : f''x^{<\omega} \subset Px\} \subset X$ *for some* $f : \lambda^{<\omega} \to P_{\omega_1}\lambda$.

(5) $\{x : x < (\lambda; f_i)_{i<\omega})\} \subset X$ *for some structure* $(\lambda; f_i)_{i<\omega}$ *with the universe*

λ and finitary functions $\{f_i : i < \omega\}$ on λ, where "$<$" denotes an elementary substructure.

Proof. We show only equivalence of (1) and (2). The other statements are equivalent to (2) by standard arguments.

Define $f : \lambda \to P_\omega\lambda$ by $f(\alpha) = \{0, \alpha + 1\}$. Then $\{x : f''x \subset Px\} \subset \{x : \omega \subset x\} \in CF^\omega_{\kappa\lambda}$. Hence $CF^\omega_{\kappa\lambda}$ is ω_1-complete by the standard argument using normality of $CF^\omega_{\kappa\lambda}$. Thus $CF^{\omega_1}_{\kappa\lambda} \subset CF^\omega_{\kappa\lambda}$ by minimality of $CF^{\omega_1}_{\kappa\lambda}$. □

2. The case $\mu = \omega_1$

In this section, we try to find the simplest form of generators of $CF^{\omega_1}_{\kappa\lambda}$ and $SCF^{\omega_1}_{\kappa\lambda}$. It is easily seen by the methods in the last section that $CF^{\omega_1}_{\kappa\lambda}$ can be characterized by two binary functions on λ. A closer coding yields that *one* binary function suffices.

2.1. **Theorem.** *$X \in CF^{\omega_1}_{\kappa\lambda}$ iff $\{x : f''x^2 \subset x\} \subset X \subset P_\kappa\lambda$ for some $f : \lambda^2 \to \lambda$.*

Proof. We show the only-if part.

Fix $f : \lambda^2 \to P_\omega\lambda$. By Proposition 1.3, it suffices to define $p : \lambda^2 \to \lambda$ and show $\{x : p''x^2 \subset x\} \subset \{x : f''x^2 \subset Px\}$.

First fix $\rho : \lambda^2 \to \lambda$ such that $\rho|\{(\alpha, \beta) \in \lambda^2 : \alpha \neq \beta\}$ is a bijection to $\lambda - \{0\}$, $\rho(\alpha, \alpha) = 0$ for $\alpha < \lambda$, $\rho(\alpha, 0)$ is a limit ordinal for $0 < \alpha < \lambda$, and $\rho(\alpha, \beta) \geq \omega$ for $\alpha < \lambda$ and $\beta < \omega$ with $\alpha \neq \beta$.

Define $g : \lambda \to P_\omega\lambda$ by $g(0) = f(0, 0)$ and $g(\gamma) = f(\rho^{-1}(\gamma)) \cup f(\gamma, \gamma)$ for $\gamma > 0$. We show $\{x : \rho''x^2 \subset x \text{ and } g''x \subset Px\} \subset \{x : f''x^2 \subset Px\}$.

Let $x \in P_\kappa\lambda$ be closed under ρ and g. Fix $\alpha, \beta \in x$. Then $f(\alpha, \beta) \subset x$, since $f(\alpha, \beta) \subset g(\rho(\alpha, \beta))$ for $\alpha \neq \beta$, and since $f(\alpha, \alpha) \subset g(\alpha)$.

For $\alpha < \lambda$, we can fix $n_\alpha < \omega$ such that $n \neq \alpha$ and $\rho(\alpha, n) \notin g(\alpha)$ for any $n_\alpha \leq n < \omega$, since $\{n < \omega : \rho(\alpha, n) \in g(\alpha)\}$ is finite. Define $h : \lambda^2 \to \lambda$ by $h(\alpha, 0) = \alpha + 1$, $\{h(\alpha, n) : n_\alpha \leq n < n_\alpha + |g(\alpha)|\} = g(\alpha)$ and $h(\alpha, \beta) = 0$ for $0 < \beta < n_\alpha$ and $n_\alpha + |g(\alpha)| \leq \beta$. We show $\{x : h''x^2 \subset x\} \subset \{x : g''x \subset Px\}$.

Let $x \in P_\kappa\lambda$ be closed under h. Fix $\alpha \in x$. Then $\omega \subset x$, since $0 = h(\alpha, \alpha)$ for $\alpha > 0$ and $n + 1 = h(n, 0)$ for $n < \omega$. Thus $g(\alpha) \subset \{h(\alpha, n) : n < \omega\} \subset x$.

Define $k : \lambda \to \lambda$ by $k(0) = h(0, 0)$ and $k(\gamma) = h(\rho^{-1}(\gamma))$ for $\gamma > 0$. We show $\{x : \rho''x^2 \subset x \text{ and } k''x \subset x\} \subset \{x : h''x^2 \subset x\}$.

Let $x \in P_\kappa\lambda$ be closed under ρ and k. Fix $\alpha, \beta \in x$. Then $h(\alpha, \beta) \in x$, since $h(\alpha, \beta) = k(\rho(\alpha, \beta))$ for $\alpha \neq \beta$, since $h(\alpha, \alpha) = \rho(\alpha, \alpha) = 0$ for $\alpha > 0$ and since $h(0, 0) = k(0)$.

Define $p : \lambda^2 \to \lambda$ by $p(\alpha, \alpha) = k(\alpha)$ and $p(\alpha, \beta) = \rho(\alpha, \beta)$ for $\alpha \neq \beta$. We show $\{x : p''x^2 \subset x\} \subset \{x : \rho''x^2 \subset x \text{ and } k''x \subset x\}$.

Let $x \in P_\kappa\lambda$ be closed under p. Fix $\alpha \in x$. Then $k(\alpha) = p(\alpha, \alpha) \in x$ and $\rho(\alpha, \beta) = p(\alpha, \beta) \in x$ for $\beta \in x$ with $\alpha \neq \beta$. We show $\rho(\alpha, \alpha) = 0 \in x$.

We can assume $\alpha > 0$. Put $\delta = h(\beta, \gamma)$, where $\rho(\beta, \gamma) = \alpha$ and $\beta \neq \gamma$. Then $\delta = k(\alpha) \in x$ and $\delta \neq \alpha$, since $h(\beta, \gamma) \neq \rho(\beta, \gamma)$ by the definition of h and ρ. Put $\eta = \rho(\delta, \alpha)$. Then $\eta > 0$ and $\eta = p(\delta, \alpha) \in x$, since $\delta \neq \alpha$. Hence by the definition of h, $0 = h(\delta, \alpha) = k(\eta) \in x$ when $\alpha \geq \omega$. When $\alpha < \omega$, we have $\eta \geq \omega$ by the definition of ρ. Put $\theta = \rho(\alpha, \eta)$. Then $\theta > 0$ and $\theta = p(\alpha, \eta) \in x$, since $\alpha \neq \eta$. Hence by the definition of h, $0 = h(\alpha, \eta) = k(\theta) \in x$, since $\eta \geq \omega$. □

2.2. **Corollary** [7]. (1) $X \in CF^{\mu}_{\kappa\lambda}$ *iff* $\{x : x \cap \mu \in \mu$ *and* $f''x^2 \subset x\} \subset X \subset P_\kappa\lambda$ *for some* $f : \lambda^2 \to \lambda$.

(2) $X \in CF_{\omega_1\lambda}$ *iff* $\{x : f''x^2 \subset x\} \subset X \subset P_{\omega_1}\lambda$ *for some* $f : \lambda^2 \to \lambda$.

We need to introduce the successor function $\sigma : \lambda \to \lambda$ to get an analog of Theorem 2.1 for $SCF^{\omega_1}_{\kappa\lambda}$.

2.3. **Theorem.** $X \in SCF^{\omega_1}_{\kappa\lambda}$ *iff* $\{x : \sigma''x \subset x$ *and* $f''x \subset x\} \subset X \subset P_\kappa\lambda$ *for some* $f : \lambda \to \lambda$.

Proof. We show the only-if part.

Fix $f : \lambda \to P_{\omega_1}\lambda$. Define $g : \lambda \to \lambda$ by $g(\omega \cdot \alpha + n) = \omega \cdot \alpha$ for odd $n < \omega$ and $\{g(\omega \cdot \alpha + n) : n < \omega \text{ is even}\} = \bigcup_{m<\omega} f(\omega \cdot \alpha + m)$. We show $\{x : \sigma''x \subset x \text{ and } g''x \subset x\} \subset \{x : f''x \subset Px\}$.

Let $x \in P_\kappa\lambda$ be closed under σ and g. Fix $\omega \cdot \alpha + l \in x$ with $l < \omega$. Then $\omega \cdot \alpha \in x$, since $\omega \cdot \alpha = g(\omega \cdot \alpha + l)$ or $g(\sigma(\omega \cdot \alpha + l))$ according to whether l is odd or even. Hence $\{\omega \cdot \alpha + n : n < \omega\} \subset x$, since x is closed under σ. Thus $f(\omega \cdot \alpha + l) \subset \bigcup_{m<\omega} f(\omega \cdot \alpha + m) \subset \{g(\omega \cdot \alpha + n) : n < \omega\} \subset x$. □

2.4. **Corollary.** (1) $X \in SCF_{\omega_1\lambda}$ *iff* $\{x : \sigma''x \subset x$ *and* $f''x \subset x\} \subset X \subset P_{\omega_1}\lambda$ *for some* $f : \lambda \to \lambda$.

(2) $SCF^{\omega_1}_{\kappa\lambda} = SCF^{\omega}_{\kappa\lambda}$.

A simple observation yields that Theorem 2.3 is optimal with respect to the number of characterizing functions.

2.5. **Proposition.** $\{x : f''x \subset x\} \not\subset \{x : \omega \subset x\}$ *for any* $f : \lambda \to \lambda$.

Proof. Let $f : \lambda \to \lambda$ be a counterexample. Set $x = \{f^m(0) : 0 < m < \omega\}$, where $f^m : \lambda \to \lambda$ is the m-fold iteration of f. Then $\omega \subset x$, since x is closed under f and by the choice of f. Pick $0 < n < \omega$ with $f^n(0) = 0$. Then $x = \{f^m(0) : 0 < m \leq n\}$ is finite, contradicting to $\omega \subset x$. □

3. The case $\mu > \omega_1$

We have already seen in the last section that $CF^{\omega_1}_{\kappa\lambda} = CF^{\omega}_{\kappa\lambda}$ and $SCF^{\omega_1}_{\kappa\lambda} = SCF^{\omega}_{\kappa\lambda}$. This never happens for the pair (ν, μ) in place of (ω_1, ω). Instead we have four kinds of simply definable subsets of $P_\kappa\lambda$ which witness the difference between $CF^{\nu}_{\kappa\lambda}$ (resp. $SCF^{\nu}_{\kappa\lambda}$) and $CF^{\mu}_{\kappa\lambda}$ (resp. $SCF^{\mu}_{\kappa\lambda}$) as a candidate for a generator of the former over the latter.

The same argument as in Theorem 2.3 settles the first case where $\nu = \xi^+$ and the candidate is $\{x : \sigma''_\xi x \subset Px\}$.

3.1. **Theorem.** $X \in SCF^{\xi^+}_{\kappa\lambda}$ *iff* $\{x : \sigma''_\xi x \subset Px$ *and* $f''x \subset x\} \subset X \subset P_\kappa\lambda$ *for some* $f : \lambda \to \lambda$.

3.2. **Corollary.** (1) $SCF^{\xi^+}_{\kappa\lambda} = SCF^{\omega}_{\kappa\lambda} + \{x : \sigma''_\xi x \subset Px\}$.

(2) $CF^{\xi^+}_{\kappa\lambda} = CF^{\omega}_{\kappa\lambda} + \{x : \sigma''_\xi x \subset Px\}$.

We have already seen in Proposition 1.2(4) the second case for $CF^{\mu}_{\kappa\lambda}$ where the candidate is $\{x : x \cap \nu \in \nu\}$. An analog of Proposition 1.2(4) fails for $SCF^{\mu}_{\kappa\lambda}$ as in Corollary 3.4(4). We have in fact the following stronger result.

3.3. **Theorem.** $SCF^{\mu^+}_{\kappa\lambda} \not\subset SCF^{\mu}_{\kappa\lambda} + \{x : x \cap \kappa \in \kappa\}$.

Proof. We show $\{x : \sigma''_\mu x \subset Px\} \notin SCF^{\mu}_{\kappa\lambda} + \{x : x \cap \kappa \in \kappa\}$.

Otherwise pick $f : \lambda \to P_\mu\lambda$ with $\{x : x \cap \kappa \in \kappa$ and $f''x \subset Px\} \subset \{x : \sigma''_\mu x \subset Px\}$.

Let $x \in [\lambda]^\kappa$ be the closure of κ under f. Then we can pick $\beta = \mu \cdot \alpha < \lambda$ with $\sigma_\mu(\beta) \cap x = 0$, since λ is the disjoint union of $\{\sigma_\mu(\mu \cdot \alpha) : \alpha < \lambda\}$. Let $y \in P_\mu\lambda$ be the closure of $\{\beta\}$ under f. Set $z = \bigcup_{n<\omega} z_n$, where $z_n \in P_\kappa\lambda$ is defined inductively by $z_0 = \{\beta\}$ and $z_{n+1} = \sup(z_n \cap \kappa) \cup \bigcup f''z_n$. Then $\sigma_\mu(\beta) \subset z$, since $z \cap \kappa \in \kappa$ and z is closed under f and by the choice of f. On the other hand $z \subset x \cup y$, since $z_n \subset x \cup y$ by induction on $n < \omega$. Hence $\sigma_\mu(\beta) \subset y$, since $\sigma_\mu(\beta) \cap x = 0$, contradicting to $y \in P_\mu\lambda$. □

3.4. **Corollary.** *Let* $\mu^+ < \nu$ *in* (1) *and* $\mu < \xi$ *in* (2) *and* (3).

(1) $SCF^{\mu^+}_{\kappa\lambda} + \{x : x \cap \nu \in \nu\} \supsetneq SCF^{\mu}_{\kappa\lambda} + \{x : x \cap \nu \in \nu\}$.
(2) $SCF^{\mu^+}_{\kappa\lambda} + \{x : \xi \subset x\} \supsetneq SCF^{\mu}_{\kappa\lambda} + \{x : \xi \subset x\}$.
(3) $SCF^{\mu^+}_{\kappa\lambda} + \{x : |x| \geq \xi\} \supsetneq SCF^{\mu}_{\kappa\lambda} + \{x : |x| \geq \xi\}$.
(4) $SCF^{\nu}_{\kappa\lambda} \supsetneq SCF^{\mu}_{\kappa\lambda} + \{x : x \cap \nu \in \nu\}$.
(5) $SCF^{\xi^+}_{\kappa\lambda} \supsetneq SCF^{\mu}_{\kappa\lambda} + \{x : \xi \subset x\}$.
(6) $SCF^{\xi^+}_{\kappa\lambda} \supsetneq SCF^{\mu}_{\kappa\lambda} + \{x : |x| \geq \xi\}$.

The same argument as in Proposition 1.3 settles the third case for $CF^{\mu}_{\kappa\lambda}$ where $\nu = \xi^+$ and the candidate is $\{x : \xi \subset x\}$.

3.5. **Proposition.** $CF^{\xi^+}_{\kappa\lambda} = CF^{\omega}_{\kappa\lambda} + \{x : \xi \subset x\}$.

An analog of Proposition 3.5 fails for $SCF^{\mu}_{\kappa\lambda}$ as in Corollary 3.4(5). We have in fact the following stronger result.

3.6. **Theorem.** *Let* $\mu \leq \xi$. *Then* $SCF^{\mu}_{\kappa\lambda} + \{x : x \cap \xi^+ \in \xi^+\} \supsetneq SCF^{\mu}_{\kappa\lambda} + \{x : \xi \subset x\}$.

Proof. We show $\{x : x \cap \xi^+ \in \xi^+\} \notin SCF^{\mu}_{\kappa\lambda} + \{x : \xi \subset x\}$.

Otherwise pick $f : \lambda \to P_\mu\lambda$ with $\{x : \xi \subset x$ and $f''x \subset Px\} \subset \{x : x \cap \xi^+ \in \xi^+\}$.

Let $x \in [\lambda]^\xi$ be the closure of ξ under f. Set $\beta = x \cap \xi^+ \in \xi^+$. Let $y \in P_\mu\lambda$ be the closure of $\{\beta + \xi\}$ under f. Then $\beta + \xi \in (x \cup y) \cap \xi^+$ is an ordinal, since $\xi \subset x \cup y$ is closed under f and by the choice of f. Hence $\beta + \xi \subset x \cup y$. Thus $\{\beta + \gamma : \gamma < \xi\} \subset y$, since $x \cap (\beta + \xi) \subset \beta$, contradicting to $y \in P_\mu\lambda$. □

We need to generalize the notion of square bracket partition relations (see [3]) to consider the forth case for $CF^{\mu}_{\kappa\lambda}$ where $\nu = \xi^+$ and the candidate is $\{x : |x| \geq \xi\}$.

3.7. **Theorem.** *Let* $\mu \leq \zeta \leq \xi$. *Then* $CF^{\zeta^+}_{\kappa\lambda} \subset CF^{\mu}_{\kappa\lambda} + \{x : |x| \geq \xi\}$ *iff* $\lambda \not\to [\xi]^{<\omega}_{P_\mu\zeta}$.

Proof. First observe that $CF^{\zeta^+}_{\kappa\lambda} \subset CF^{\mu}_{\kappa\lambda} + \{x : |x| \geq \xi\}$ iff $\{x : \zeta \subset x\} \in CF^{\mu}_{\kappa\lambda} + \{x : |x| \geq \xi\}$ iff there exists $f : \lambda^2 \to P_\mu\lambda$ with $\{x : |x| \geq \nu$ and $f''x^2 \subset Px\} \subset \{x : \zeta \subset x\}$ iff there exists $f : \lambda^2 \to P_\mu\lambda$ with $\zeta \subset x_y$ for any $y \in [\lambda]^\xi$, where

$x_z \in P_\kappa\lambda$ is defined to be the closure of $z \in P_\kappa\lambda$ under f iff $(*)$ there exists $f : \lambda^2 \to P_\mu\lambda$ with $\bigcup g''[y]^{<\omega} = \zeta$ for any $y \in [\lambda]^\xi$, where $g : [\lambda]^{<\omega} \to P_\mu\zeta$ is defined by $g(s) = x_s \cap \zeta$.

The last equivalence follows from the fact that $x_z = \bigcup_{s\in[z]^{<\omega}} x_s$ for any $z \in P_\kappa\lambda$, which we show.

Define $x_{z,n} \in P_\kappa\lambda$ inductively by $x_{z,0} = z$ and $x_{z,n+1} = x_{z,n} \cup \bigcup f''x^2_{z,n}$. Then $x_z = \bigcup_{n<\omega} x_{z,n}$ and $x_{z,n} \subset \bigcup_{s\in[z]^{<\omega}} x_s$ by induction on $n < \omega$.

Now we show that $(*)$ holds iff $\lambda \not\to [\xi]^{<\omega}_{P_\mu\zeta}$.

If part. Let $h : [\lambda]^{<\omega} \to P_\mu\zeta$ witness $\lambda \not\to [\xi]^{<\omega}_{P_\mu\zeta}$. Define $f : \lambda^2 \to P_\mu\lambda$ by $f(\alpha, \beta) = \{\pi(\alpha, \beta)\} \cup \bigcup h''[y_\alpha]^{<\omega}$, where $y_\alpha \in P_{\omega_1}\lambda$ is the closure of $\{\alpha\}$ under π^{-1}. We show that $h(s) \subset x_s$ for any $s \in [\lambda]^{<\omega} - \{0\}$, which immediately implies that f witnesses $(*)$.

Fix $s = \{\alpha_i : i < n\} \in [\lambda]^{<\omega}$. Define $\beta_i < \lambda$ inductively for $i < n$ by $\beta_0 = \alpha_0$ and $\beta_{i+1} = \pi(\beta_i, \alpha_{i+1})$. Then $\beta_i \in x_s$ by induction on $i < n$, since x_s is closed under π. Thus $h(s) \subset f(\beta_n, \beta_n) \subset x_s$, since $s \in [y_{\beta_n}]^{<\omega}$.

Only-if part. $g : [\lambda]^{<\omega} \to P_\mu\zeta$ defined as in $(*)$ clearly witnesses $\lambda \not\to [\xi]^{<\omega}_{P_\mu\zeta}$. □

It is easily seen that $\lambda \to [\xi]^{<\omega}_{P_\mu\mu}$ iff $\lambda \to [\xi]^{<\omega}_{\mu,<\mu}$ a general form of Chang's conjecture (see [3]).

3.8. **Corollary.** *Let $\mu \le \xi$. Then*

(1) $CF^{\mu^+}_{\kappa\lambda} + \{x : |x| \ge \xi\} = CF^{\mu}_{\kappa\lambda} + \{x : |x| \ge \xi\}$ *iff* $\lambda \not\to [\xi]^{<\omega}_{\mu,<\mu}$.

(2) $CF^{\xi^+}_{\kappa\lambda} = CF^{\mu}_{\kappa\lambda} + \{x : |x| \ge \xi\}$ *iff* $\lambda \not\to [\xi]^{<\omega}_{P_\mu\xi}$.

An analog of Corollary 3.8(2) fails for $SCF^{\mu}_{\kappa\lambda}$ regardless of partition relations as in Corollary 3.4(6). We have in fact the following stronger result.

3.9. **Theorem.** *Let $\mu \le \xi$. Then $SCF^{\mu}_{\kappa\lambda} + \{x : \xi \subset x\} \supsetneq SCF^{\mu}_{\kappa\lambda} + \{x : |x| \ge \xi\}$.*

Proof. We show $\{x : \xi \subset x\} \notin SCF^{\mu}_{\kappa\lambda} + \{x : |x| \ge \xi\}$.

Otherwise pick $f : \lambda \to P_\mu\lambda$ with $\{x : |x| \ge \xi$ and $f''x \subset Px\} \subset \{x : \xi \subset x\}$.

Let $x_\alpha \in P_\mu\lambda$ be the closure of $\{\alpha\}$ under f for $\alpha < \lambda$. Define $g : \lambda \to \mu$ by $g(\alpha) = \sup(x_\alpha \cap \mu) + 1$. Pick $y \in [\lambda]^\xi$ and $\gamma < \mu$ with $g''y = \{\gamma\}$. Then $z = \bigcup_{\alpha\in y} x_\alpha \in [\lambda]^\xi$ is closed under f and $z \cap \mu \subset \bigcup g''y \subset \gamma$, contradicting to the choice of f. □

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