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TWISTING OPERATIONS AND COMPOSITE KNOTS

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ABSTRACT. Suppose that a composite knot K in S^3 can be changed to a trivial knot by $1/n$-surgery along a trivial loop C. We show that $|n| \leq 2$. Moreover, if there is a decomposing sphere of K which meets C in two points, then $|n| \leq 1$.

1. INTRODUCTION

Let K be a knot in the 3-sphere S^3 and D a disk which intersects K transversely in its interior. Let $C = \partial D$. We get a new knot K^* in S^3 as the image of K after doing $1/n$-surgery along C. We say that K^* is obtained from K by *n-twisting* along C. In particular, this operation is called a *trivializing n-twist* of K if K^* is unknotted. We remark that a crossing change is equivalent to ± 1-twist on a disk which intersects K in precisely two points.

In [4], Mathieu asked if there is a composite knot which admits a trivializing twist. Several families of composite knots are known to admit trivializing twists at present [5], [7], [11]. Since all the examples of trivializing twists of composite knots are ± 1-twists, it is conjectured that if a composite knot admits a trivializing n-twist, then $|n| \leq 1$ [6]. In fact, Motegi [6] proved that $|n| \leq 5$, by making use of Gordon's result about Dehn fillings on hyperbolic manifolds [2].

In this paper we improve Motegi's result as follows.

Theorem 1. *If a composite knot admits a trivializing n-twist, then $|n| \leq 2$.*

The possibility of $|n| = 2$ remains an open problem.

If a knot K is composite, then there is a 2-sphere S which intersects K transversely in two points, such that each one of the 3-balls bounded by S intersects K in a knotted spanning arc. Such a sphere is called a *decomposing sphere* of K.

Theorem 2. *Suppose that a composite knot K admits a trivializing n-twist along C and that there is a decomposing sphere S of K which intersects C transversely in two points. Then $|n| \leq 1$.*

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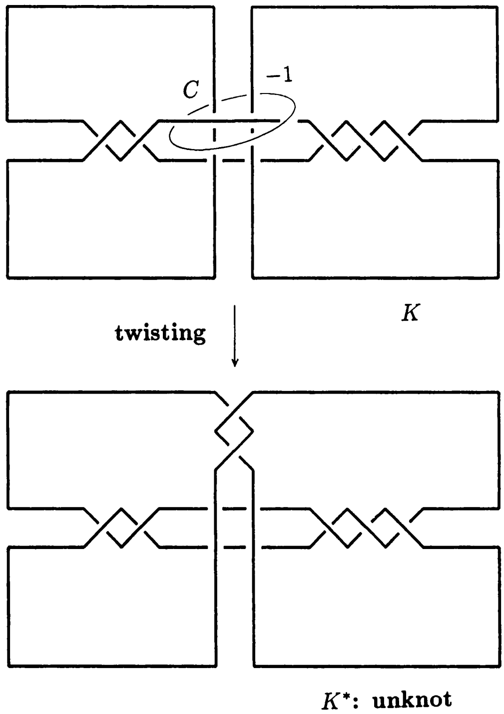

FIGURE 1

It is easy to verify that all known examples in the above papers satisfy this assumption. An example is illustrated in Figure 1.

Scharlemann [9] proved that unknotting number one knots are prime. That is, a composite knot cannot be trivialized by ± 1-twists on a disk which meets the knot in two points. (See also [10].) Miyazaki-Yasuhara [5] found many examples of composite knots which do not admit trivializing twists.

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2. PRELIMINARIES

Let K be a composite knot in S^3. Suppose that K admits a trivializing n-twist along C. Let $M = S^3 - \operatorname{Int} N(K \cup C)$. Let us write $T = \partial N(C)$, $T' = \partial N(K)$. Slopes on T or T' will be parametrized by $\mathscr{Q} \cup \{1/0\}$ in the usual way (cf. [8]), using a meridian-longitude basis. Since $K \cup C$ is unsplittable in S^3, M is irreducible. For any slope r on T, let $M(r)$ denote the manifold obtained from M by r-Dehn filling on T, that is, by attaching a solid torus J to M along T so that r bounds a disk in J. It is immediate from the definitions that K is trivialized by n-twisting along C if and only if $M(1/n)$ is a solid torus. Note that $M(1/0) = S^3 - \operatorname{Int} N(K)$.

Let S be a decomposing sphere of K. Isotope S so that $S \cap N(C)$ is a disjoint union of meridian disks of $N(C)$ and $m = |S \cap N(C)|$ is minimal. Note that $m \geq 2$. Then $P = S \cap M$ is an incompressible planar surface in M, with two *outer boundary components* $\partial_0 P$, $\partial_\infty P$, lying in T', and m *inner boundary components* $\partial_i P$, $i = 1, \dots, m$, lying in T. Here, the inner boundary components are numbered so that they are consecutive on T. Each component of ∂P has slope $1/0$ in T or T'.

Let D_0 be a meridian disk of $M(1/n)$. Isotope D_0 so that $D_0 \cap J$ is a disjoint union of meridian disks of J. We choose D_0 so that $l = |D_0 \cap J|$ is minimal over all meridian disks of $M(1/n)$. Note that $l \geq 2$. If $l = 1$, then we regard J as a regular neighborhood of a core of $M(1/n)$. This would imply that $M(1/0)$ is a solid torus. From the minimality of l, $Q = D_0 \cap M$ is an incompressible planar surface in M, with one outer boundary component $\partial_0 Q$, lying in T', and l inner boundary components $\partial_j Q$, $j = 1, \dots, l$, each having slope $1/n$ in T. The inner boundary components are numbered consecutively on T. It is easy to see that $\partial_0 Q$ has slope $n\omega^2/1$, where $\omega = lk(K, C)$.

By an isotopy of Q, we may assume that P and Q intersect transversely, and each outer boundary component of P intersects $\partial_0 Q$ exactly once, and each inner boundary component of P intersects each inner boundary of Q in $|n|$ points. Thus, for example, when we go around an inner boundary component of P, we will consecutively meet $\partial_1 Q, \partial_2 Q, \dots, \partial_l Q, \dots, \partial_1 Q, \dots, \partial_l Q$ (repeated $|n|$ times). By an innermost argument, we can assume that no loop component of $P \cap Q$ bounds a disk in P or Q, since P and Q are incompressible and M is irreducible.

As in [1], we form the associated graphs G_P and G_Q. Let A be the annulus obtained by capping off the inner boundary components of P by meridian disks of $N(C)$. We obtain a graph G_P in A by taking as the "fat" vertices of G_P the disks in $N(C)$ that cap off the inner boundary components of P, and as the edges of G_P the arc components of $P \cap Q$ in P. Similarly we obtain the graph G_Q in the disk D_0.

Let G denote either G_P or G_Q.

If an edge e connects a vertex to a vertex, then e is an *interior edge*; otherwise, it is a *boundary edge.* Note that G has at most two boundary edges. If G_P has two boundary edges, so does G_Q, and vice versa. Each vertex of G_P (G_Q) has degree $|n|l$ ($|n|m$, resp.).

Let e be an edge of G_P. If an end point of e is in $\partial_i P \cap \partial_j Q$, then we give this end point of e the label j. Thus each incidence of an edge of G_P at a vertex of G_P is labeled with a vertex of G_Q. Similarly in G_Q, label the end points of edges incident to vertices.

Two vertices of G_P (G_Q) are *parallel* if the corresponding inner boundary components of $P(Q)$, when given the orientations induced by some orientation of $P(Q)$, are homologous in T; otherwise, they are *antiparallel.* Since M is orientable, we have the *parity rule*:

An interior edge e of G_P connects parallel vertices in G_P if and only if e connects antiparallel vertices in G_Q.

An *x-cycle* in G is a cycle σ of edges in G such that all the vertices of G in σ are parallel and σ can be oriented so that the tail of each edge has label x. A *Scharlemann cycle* in G is an x-cycle σ in G for some label x such that σ bounds a disk face of G. In particular, a Scharlemann cycle of length 1 will be called a *trivial loop.*

Lemma 1. *G contains no trivial loops.*

Proof. This follows immediately from the minimality of l or m.

Lemma 2. *G contains no Scharlemann cycles.*

The proof is analogous to [1, proof of Lemma 2.5.2] or [3, proof of Lemma 3.3]. We omit the details.

3. Proofs

To find Scharlemann cycles, we consider the following conditions as in [1]:

($*$) There exists a vertex x of G such that for each label y there is an edge of G incident to x with label y, connecting x to an antiparallel vertex of G.

($**$) For each vertex x of G there exists a label $y(x)$ such that each edge of G incident to x with label $y(x)$ connects x either to a parallel vertex of G or to an outer boundary.

In fact, ($**$) is the negation of ($*$).

Lemma 3. *Suppose that G_P satisfies ($*$). Then G_Q contains a Scharlemann cycle.*

Proof. See [1, Lemmas 2.6.2 and 2.6.3].

Remark. In general, we cannot exchange the roles of P and Q in the statement of Lemma 3. Because an x-cycle in G_P does not necessarily bound a disk in the annulus A. However, when G_P has only one boundary edge, we can conclude that G_P contains a Scharlemann cycle if G_Q satisfies ($*$).

Lemma 4. *Let x be a vertex of G_P. If there exist successive l edges of G_P connecting x to antiparallel vertices, then G_Q contains a Scharlemann cycle.*

Proof. This follows immediately from Lemma 3.

Lemma 5. *If G contains a parallel family of edges connecting parallel vertices, then either the sets of labels at the two ends of the family are disjoint, or G contains a Scharlemann cycle. In particular, if G_P (G_Q) contains a parallel family of more than $l/2$ ($m/2$, resp.) edges connecting parallel vertices, then G_P (G_Q) contains a Scharlemann cycle.*

Proof. See [1, Lemma 2.6.6 and Corollary 2.6.7].

Lemma 6. *Suppose that $|n| \geq 2$ and that G_Q satisfies ($**$). Then either G_Q contains a Scharlemann cycle or every vertex of G_P belongs to a boundary edge of G_P.*

Proof. This is essentially [1, Lemma 2.6.4]. The proof works well even if G_P is a graph in an annulus.

We remark that the latter conclusion of Lemma 6 implies that G_P has exactly two vertices.

Now suppose that G_P satisfies ($**$). Let v be a vertex of G_P. There exists a label $y(v)$ such that each one of $|n|$ edges of G_P incident to v with label $y(v)$ connects v either to a parallel vertex or to ∂A. Fix the label $y(v)$. These $|n|$ edges will be called the *$y(v)$-edges* at v. A *corner* at v is an interval on the boundary of the fat vertex v between successive labels $y(v)$. There are $|n|$ corners around v, and there are $l-1$ incidences of edges to v in the interior of a corner. Let $\Gamma = G_P - \{\text{boundary edges}\}$. Let $\overline{\Gamma}$ be the reduced graph of Γ, obtained by amalgamating all mutually parallel edges in the obvious way. Then G_P, Γ, and $\overline{\Gamma}$ have the same vertex set.

We now want to estimate the degree $\deg_{\overline{\Gamma}}(v)$ of v in $\overline{\Gamma}$.

Lemma 7. *Suppose that G_P satisfies $(**)$. Let v be a vertex of G_P, and let $b(v)$ be the number of boundary edges incident to v. Then $\deg_{\overline{\Gamma}}(v) \geq 2|n| - b(v)$.*

Proof. By Lemmas 2 and 5, any pair of $y(v)$-edges is not parallel. Hence the $y(v)$-edges, except for boundary edges, correspond to distinct edges of $\overline{\Gamma}$. Also, not all the $l-1$ edges incident to v in the interior of a corner are parallel to $y(v)$-edges. Therefore, the interior of a corner yields at least one edge of $\overline{\Gamma}$ unless it does not meet a boundary edge. The conclusion follows from these observations.

In fact, we have three possibilities, according to the situation in G_P:
(1) No boundary edge is incident to v.
(2) Only one boundary edge is incident to v.
(3) Two boundary edges are incident to v.
There is at most one vertex of G_P that satisfies (3). If a vertex satisfies (2), then G_P has precisely two such vertices.

The following lemma is an easy consequence of Lemma 7 and the observation above.

Lemma 8. *Let Λ be a component of $\overline{\Gamma}$. Let V and E be the number of vertices and edges of Λ. Then $|n|V \leq E+1$.*

Possibly, G_P is disconnected. Choose a point $z \in \partial A - G_P$. We define a partial ordering on the set of components of G_P as in [1]. For two components H_1 and H_2 of G_P, $H_1 < H_2$ if and only if every path in A from H_1 to z meets H_2: A component of G_P is *extremal* if it is minimal with respect to the partial ordering for some choice of z.

Proof of Theorem 1. Suppose that $|n| \geq 3$. If G_P satisfies $(*)$, then G_Q would contain a Scharlemann cycle by Lemma 3, contradicting Lemma 2. Thus G_P satisfies $(**)$.

We may assume that G_P is connected. If G_P is disconnected, we will replace G_P by an extremal component. (We avoid a component without vertex.) We consider the reduced graph $\overline{\Gamma}$ of $\Gamma = G_P - \{\text{boundary edge}\}$ as before. Since G_P is connected, $\overline{\Gamma}$ is also connected. Let V, E, and F be the number of vertices, edges, and faces of $\overline{\Gamma}$. We do not count the region meeting a component of ∂A as a face of $\overline{\Gamma}$. By Lemma 8, $3V \leq E+1$. Since $\overline{\Gamma}$ has no 1-sided faces or parallel edges, every face has at least three sides. Let F_0 and F_∞ be the components of $A - \overline{\Gamma}$ containing $\partial_0 P$ and $\partial_\infty P$ respectively (possibly, $F_0 = F_\infty$). The frontiers $\operatorname{Fr} F_0$ and $\operatorname{Fr} F_\infty$ can be expressed as the unions of a sequence of edges. Let a and b be the number of edges in $\operatorname{Fr} F_0$ and $\operatorname{Fr} F_\infty$ respectively. Note that a double edge is counted twice. Then $3F + a \leq 2E$ if $F_0 = F_\infty$, or $3F + (a+b) \leq 2E$ if $F_0 \neq F_\infty$. By Euler's formula, $1 = V - E + F \leq \frac{1-a}{3}$ if $F_0 = F_\infty$, or $0 = V - E + F \leq \frac{1-(a+b)}{3}$ if $F_0 \neq F_\infty$. This is a contradiction in either case. This completes the proof.

Proof of Theorem 2. Suppose that $|n| \geq 2$. Since $|S \cap N(C)| = 2$, G_P has exactly two vertices x and y that are antiparallel. If G_P has only one boundary edge, then the arc corresponding to the boundary edge is essential in the annulus A. If G_Q satisfies $(*)$, then G_P contains a Scharlemann cycle by the remark after Lemma 3. If G_Q satisfies $(**)$, then Lemma 6 implies that G_Q contains a Scharlemann cycle, since no vertex of G_P belongs to a boundary edge. In either

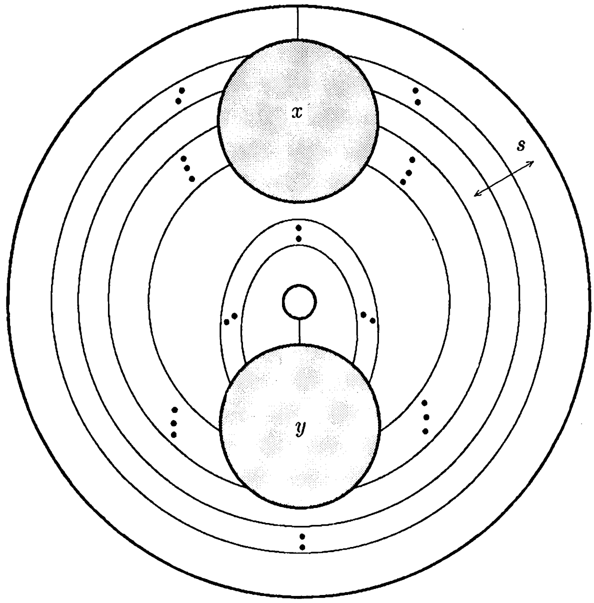

FIGURE 2

case, Lemma 2 gives a contradiction. Hence G_P has exactly two boundary edges.

If two boundary edges are incident to the same vertex x, say, then there is a loop σ at y, since x and y have the same degree. However, σ bounds a disk which does not contain the vertex x. Hence, there would be a trivial loop. This contradicts Lemma 1. Thus each vertex belongs to a single boundary edge.

We distinguish two cases.

(1) G_P contains no loops.

Then all the interior edges incident to x connect vertices x and y. By Lemma 4, G_Q contains a Scharlemann cycle. This contradicts Lemma 2.

(2) G_P contains a loop.

There is a loop based at x. Any loop must be essential in A. Consider the edge e incident to x immediately to the right of the boundary edge. Then e must be a loop. Otherwise, a loop based at x would be inessential in A. Then, without loss of generality, we have a situation as in Figure 2.

Suppose that there are s parallel loops, including e. Then by Lemma 5, $s \leq l/2$. But if $s = l/2$, then a loop has the same label at both ends, which contradicts the parity rule. Therefore, $2s + 1 \leq l$. Hence, there are at least l edges connecting x to y, since x has degree $|n|l \geq 2l$. Then, by Lemma 4, G_Q contains a Scharlemann cycle, a contradiction. This completes the proof.

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