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ON SOME MODULAR EQUATIONS OF DEGREE 5

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ABSTRACT. The main purpose of this paper is to examine certain well-known modular equations of degree 5 from a very intuitive angle. Unlike the classical treatment of this subject, which requires parametrization of certain key quantities, we will give direct proofs to these equations by recasting them as theta function identities which are quite interesting in their own right.

Ramanujan discovered an astounding number of modular equations. Since he did not supply proofs to these identities, we do not know his methods. In this paper, we will see that in the study of the modular identities of degree 5, the following well-known identities (see [1, p. 262, Entries 10(iv) and (v)]) are of great significance:

$$\vartheta_2^2(q) - \vartheta_2^2(q^5) = 4q^{\frac{1}{2}} \prod_{n=1}^{\infty} (1+q^{2n})(1+q^{5n})(1-q^{5n})^2(1+q^{10n-5}), \tag{1}$$

$$\vartheta_3^2(q) - \vartheta_3^2(q^5) = 4q \prod_{n=1}^{\infty} (1+q^{2n-1})(1+q^{5n})(1-q^{5n})^2(1+q^{10n}). \tag{2}$$

In combination with another identity (7), they yield many very striking theta function identities. In particular, in the course of our discussions, we will provide simple direct proofs to the following two identities:

$$\vartheta_3^2(q)\vartheta_3^2(q^5) - \vartheta_2^2(q)\vartheta_2^2(q^5) - \vartheta_4^2(q)\vartheta_4^2(q^5) = 8q \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{10n})^2,$$

$$\sqrt{\frac{\vartheta_2^5(q)}{\vartheta_2(q^5)}} - \sqrt{\frac{\vartheta_3^5(q)}{\vartheta_3(q^5)}} - \sqrt{\frac{\vartheta_4^5(q)}{\vartheta_4(q^5)}} = 2\sqrt{\prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^{10n})}}.$$

According to Bruce Berndt (see [1, p. 285]), direct proofs of these two identities have not been given. We assume the reader is familiar with the basic properties of the theta functions covered in [2, Chapters 21 and 22].

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To begin, we observe that from (1) and (2) and the triple product identities for the theta functions, we have

$$\frac{\vartheta_2^2(q)}{\vartheta_2^2(q^5)} - 1 = \frac{1}{q^2}\prod_{n=1}^{\infty}\frac{(1+q^{2n})}{(1+q^{10n})^5}, \tag{3}$$

$$\frac{\vartheta_3^2(q)}{\vartheta_3^2(q^5)} - 1 = 4q\prod_{n=1}^{\infty}\frac{(1+q^{2n-1})}{(1+q^{10n-5})^5} \tag{4}$$

and replacing q by $-q$ in (4) and using the fact that $\vartheta_3(-q) = \vartheta_4(q)$, we have

$$\frac{\vartheta_4^2(q)}{\vartheta_4^2(q^5)} - 1 = -4q\prod_{n=1}^{\infty}\frac{(1-q^{2n-1})}{(1-q^{10n-5})^5} = -4q\prod_{n=1}^{\infty}\frac{(1+q^{5n})^5}{(1+q^n)}. \tag{5}$$

Now multiplying (3), (4), and (5) together, we obtain

$$\left(1-\frac{\vartheta_2^2(q)}{\vartheta_2^2(q^5)}\right)\left(1-\frac{\vartheta_3^2(q)}{\vartheta_3^2(q^5)}\right)\left(1-\frac{\vartheta_4^2(q)}{\vartheta_4^2(q^5)}\right) = 16; \tag{6}$$

and applying the imaginary transformation to (6) yields

$$\left(1-\frac{5\vartheta_2^2(q^5)}{\vartheta_2^2(q)}\right)\left(1-\frac{5\vartheta_3^2(q^5)}{\vartheta_3^2(q)}\right)\left(1-\frac{5\vartheta_4^2(q^5)}{\vartheta_4^2(q)}\right) = 16. \tag{6'}$$

A very important identity which plays a vital role in the subsequent discussions of this paper is the following

$$\frac{\vartheta_3(q^5)\vartheta_4(q^5)}{\vartheta_3(q)\vartheta_4(q)} + \frac{\vartheta_2(q^5)\vartheta_3(q^5)}{\vartheta_2(q)\vartheta_3(q)} - \frac{\vartheta_2(q^5)\vartheta_4(q^5)}{\vartheta_2(q)\vartheta_4(q)} = 1. \tag{7}$$

It is written (and proved in [1, p. 276, (12.32)]) in terms of Ramanujan's notation as

$$\frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)} + q\left(\frac{\psi^2(q^5)}{\psi^2(q)} - \frac{\psi^2(-q^5)}{\psi^2(-q)}\right) = 1,$$

where $\varphi(q) =: \sum_{n=-\infty}^{\infty} q^{n^2} = \vartheta_3(q)$ and $\psi(q) =: \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{1}{2}q^{-\frac{1}{8}}\vartheta_2(q^{\frac{1}{2}})$. Since the proof is elementary, for the sake of completeness, we will reproduce it at the end of this paper. We rewrite this identity as

$$\frac{\vartheta_2(q)}{\vartheta_2(q^5)} + \frac{\vartheta_4(q)}{\vartheta_4(q^5)} - \frac{\vartheta_3(q)}{\vartheta_3(q^5)} = \frac{\vartheta_2(q)\vartheta_3(q)\vartheta_4(q)}{\vartheta_2(q^5)\vartheta_3(q^5)\vartheta_4(q^5)} = \frac{1}{q}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^3}{(1-q^{10n})^3}. \tag{7'}$$

An application of the imaginary transformation to (7′) gives

$$\frac{\vartheta_2(q^5)}{\vartheta_2(q)} + \frac{\vartheta_4(q^5)}{\vartheta_4(q)} - \frac{\vartheta_3(q^5)}{\vartheta_3(q)} = \frac{5\vartheta_2(q^5)\vartheta_3(q^5)\vartheta_4(q^5)}{\vartheta_2(q)\vartheta_3(q)\vartheta_4(q)}.$$

In a moment, we shall see how to use (7′) to establish the earlier mentioned identity

$$\vartheta_3^2(q)\vartheta_3^2(q^5) - \vartheta_2^2(q)\vartheta_2^2(q^5) - \vartheta_4^2(q)\vartheta_4^2(q^5) = 8q\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q^{10n})^2. \tag{8}$$

This identity is often written as (see [1, p. 280, Entry 13(i)])

$$\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1,$$

where $\alpha = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)}$ and $\beta = \frac{\vartheta_2^4(q^5)}{\vartheta_3^4(q^5)}$.

To this end, we need the following identity:

$$\begin{aligned}(\vartheta_2^2(q) - \vartheta_2^2(q^5))^2 \frac{\vartheta_2(q^5)}{\vartheta_2(q)} &= (\vartheta_3^2(q) - \vartheta_3^2(q^5))^2 \frac{\vartheta_3(q^5)}{\vartheta_3(q)} \\ &= (\vartheta_4^2(q) - \vartheta_4^2(q^5))^2 \frac{\vartheta_4(q^5)}{\vartheta_4(q)} = 16q^2 \prod_{n=1}^{\infty} \frac{(1-q^{10n})^5}{(1-q^{2n})}.\end{aligned} \tag{9}$$

The identity (9) follows immediately from (1) and (2); we leave its proof to the reader.

The proof of (8) goes as follows: From (7′), (9), and the well-known fact that [2, p. 467]

$$\vartheta_3^4(q) = \vartheta_2^4(q) + \vartheta_4^4(q), \tag{10}$$

we see that

$$\begin{aligned}&\vartheta_3^2(q)\vartheta_3^2(q^5) - \vartheta_2^2(q)\vartheta_2^2(q^5) - \vartheta_4^2(q)\vartheta_4^2(q^5) \\ &= \frac{1}{2}\{(\vartheta_2^2(q) - \vartheta_2^2(q^5))^2 + (\vartheta_4^2(q) - \vartheta_4^2(q^5))^2 - (\vartheta_3^2(q) - \vartheta_3^2(q^5))^2\} \\ &= \left(8q^2 \prod_{n=1}^{\infty} \frac{(1-q^{10n})^5}{(1-q^{2n})}\right)\left(\frac{\vartheta_2(q)}{\vartheta_2(q^5)} + \frac{\vartheta_4(q)}{\vartheta_4(q^5)} - \frac{\vartheta_3(q)}{\vartheta_3(q^5)}\right) \\ &= 8q \prod_{n=1}^{\infty} (1-q^{2n})^2 (1-q^{10n})^2.\end{aligned}$$

This completes the proof of (8).

We note that the identity (9) also can be written as

$$\vartheta_2^2(q) - \vartheta_2^2(q^5) = 4q \left(\prod_{n=1}^{\infty} \frac{(1-q^{10n})^5}{(1-q^{2n})}\right)^{\frac{1}{2}} \left(\frac{\vartheta_2(q)}{\vartheta_2(q^5)}\right)^{\frac{1}{2}}, \tag{11}$$

$$\vartheta_3^2(q) - \vartheta_3^2(q^5) = 4q \left(\prod_{n=1}^{\infty} \frac{(1-q^{10n})^5}{(1-q^{2n})}\right)^{\frac{1}{2}} \left(\frac{\vartheta_3(q)}{\vartheta_3(q^5)}\right)^{\frac{1}{2}}, \tag{12}$$

and

$$\vartheta_4^2(q) - \vartheta_4^2(q^5) = -4q \left(\prod_{n=1}^{\infty} \frac{(1-q^{10n})^5}{(1-q^{2n})}\right)^{\frac{1}{2}} \left(\frac{\vartheta_4(q)}{\vartheta_4(q^5)}\right)^{\frac{1}{2}}. \tag{13}$$

We now see that $(11) \times \vartheta_2^2(q) - (12) \times \vartheta_3^2(q) + (13) \times \vartheta_4^2(q)$ (and together with (10)) yields

$$\begin{aligned}&\vartheta_3^2(q)\vartheta_3^2(q^5) - \vartheta_2^2(q)\vartheta_2^2(q^5) - \vartheta_4^2(q)\vartheta_4^2(q^5) \\ &= 4q \left(\prod_{n=1}^{\infty} \frac{(1-q^{10n})^5}{(1-q^{2n})}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\vartheta_2^5(q)}{\vartheta_2(q^5)}} - \sqrt{\frac{\vartheta_3^5(q)}{\vartheta_3(q^5)}} - \sqrt{\frac{\vartheta_4^5(q)}{\vartheta_4(q^5)}}\right).\end{aligned} \tag{14}$$

From (8) and (14), we deduce the following interesting identity:

$$\sqrt{\frac{\vartheta_2^5(q)}{\vartheta_2(q^5)}}-\sqrt{\frac{\vartheta_3^5(q)}{\vartheta_3(q^5)}}-\sqrt{\frac{\vartheta_4^5(q)}{\vartheta_4(q^5)}}=2\sqrt{\prod_{n=1}^{\infty}\frac{(1-q^{2n})^5}{(1-q^{10n})}}. \tag{15}$$

Multiplying (15) by $\sqrt{\frac{\vartheta_3^5(q)}{\vartheta_3(q^5)}}$ and the using facts that $\vartheta_3(q)\vartheta_4(q)=\vartheta_4^2(q^2)$ and $\vartheta_2(q)\vartheta_3(q)=\frac{1}{2}\vartheta_2^2(q^{\frac{1}{2}})$, we have

$$\frac{1}{4}\frac{\vartheta_2^5(q^{\frac{1}{2}})}{\vartheta_2(q^{\frac{5}{2}})}-\frac{\vartheta_3^5(q)}{\vartheta_3(q^5)}-\frac{\vartheta_4^5(q^2)}{\vartheta_4(q^{10})}=2\prod_{n=1}^{\infty}\frac{(1-q^{2n})^5(1+q^{2n-1})^5}{(1-q^{10n})(1+q^{10n-5})}.$$

This identity is listed in [1, p. 285] as

$$\frac{\varphi^5(q)}{\varphi(q^5)}+\frac{\varphi^5(-q^2)}{\varphi(-q^{10})}+\frac{2f^5(q)}{f(q^5)}=4\frac{\psi^5(q)}{\psi(q^5)},$$

which, in turn, gives the modular equation (see [1, p. 280, Entry 13(ii)])

$$\left(\frac{\alpha^5}{\beta}\right)^{\frac{1}{8}}-\left(\frac{(1-a)^5}{(1-\beta)}\right)^{\frac{1}{8}}=1+2^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{\frac{1}{24}}.$$

Applying the imaginary transformation to (15), we obtain

$$\sqrt{\frac{\vartheta_4^5(q^5)}{\vartheta_4(q)}}-\sqrt{\frac{\vartheta_2^5(q^5)}{\vartheta_2(q)}}-\sqrt{\frac{\vartheta_3^5(q^5)}{\vartheta_3(q)}}=2q\sqrt{\prod_{n=1}^{\infty}\frac{(1-q^{10n})^5}{(1-q^{2n})}}. \tag{15'}$$

On the other hand, if we consider $(11)\times\vartheta_2^2(q^5)-(12)\times\vartheta_3^2(q^5)+(13)\times\vartheta_4^2(q^5)$ and (10), (14), and (15) we have

$$\sqrt{\vartheta_4(q)\vartheta_4^3(q^5)}+\sqrt{\vartheta_3(q)\vartheta_3^3(q^5)}-\sqrt{\vartheta_2(q)\vartheta_2^3(q^5)}=2\sqrt{\prod_{n=1}^{\infty}\frac{(1-q^{2n})^5}{(1-q^{10n})}}. \tag{16}$$

Applying Jacobi's imaginary transformation to (16), we obtain

$$\sqrt{\vartheta_2(q^5)\vartheta_2^3(q)}+\sqrt{\vartheta_3(q^5)\vartheta_3^3(q)}-\sqrt{\vartheta_4(q^5)\vartheta_4^3(q)}=10q\sqrt{\prod_{1}^{\infty}\frac{(1-q^{10n})^5}{(1-q^{2n})}}. \tag{16'}$$

We now reproduce Berndt's proof of identity (7). First, we observe that (7) is equivalent to

$$\left(1-\frac{\vartheta_2(q^5)\vartheta_3(q^5)}{\vartheta_2(q)\vartheta_3(q)}\right)\left(1+\frac{\vartheta_2(q^5)\vartheta_4(q^5)}{\vartheta_2(q)\vartheta_4(q)}\right)=\left(1-\frac{\vartheta_2^2(q^5)}{\vartheta_2^2(q)}\right)\frac{\vartheta_3(q^5)\vartheta_4(q^5)}{\vartheta_3(q)\vartheta_4(q)}. \tag{17}$$

To prove (17), we let

$$h(q)=1-\frac{\vartheta_2(q^5)\vartheta_3(q^5)}{\vartheta_2(q)\vartheta_2(q)}$$

which is the first factor in the left-hand side of (17). Since $\vartheta_3(-q)=\vartheta_4(q)$ and

$$\frac{\vartheta_2(q^5)}{\vartheta_2(q)}=q\prod_{n=1}^{\infty}\frac{(1-q^{10n})(1+q^{10n})^2}{(1-q^{2n})(1+q^{2n})^2},$$

we see that

$$h(-q) = 1 + \frac{\vartheta_2(q^5)\vartheta_4(q^5)}{\vartheta_2(q)\vartheta_4(q)}$$

which is exactly the second factor in the left-hand side of (17). Now recalling the well-known fact that $\vartheta_3(q)\vartheta_4(q) = \vartheta_4^2(q^2)$, we see that (17) becomes

$$h(q)h(-q) = h(q^2)\frac{\vartheta_4^2(q^{10})}{\vartheta_4^2(q^2)}. \tag{18}$$

Using (1) and the fact that $\vartheta_2(q)\vartheta_3(q) = \frac{1}{2}\vartheta_2^2(q^{\frac{1}{2}})$, we see that

$$\begin{aligned} h(q) &= 1 - \frac{\vartheta_2^2(q^{\frac{5}{2}})}{\vartheta_2^2(q^{\frac{1}{2}})} = \prod_{n=1}^{\infty} \frac{(1-q^{5n})^2}{(1+q^{5n})(1-q^n)^2(1+q^n)^3} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^{10n})^2(1-q^{10n-5})^3}{(1-q^{2n})^2(1+q^{2n})(1+q^{2n-1})}. \end{aligned}$$

Thus

$$h(-q) = \prod_{n=1}^{\infty} \frac{(1-q^{10n})^2(1+q^{10n-5})^3}{(1-q^{2n})^2(1+q^{2n})(1-q^{2n-1})}.$$

Hence

$$\begin{aligned} h(q)h(-q) &= \prod_{n=1}^{\infty} \frac{(1-q^{10n})^4(1-q^{20n-10})^3}{(1-q^{2n})^4(1+q^{2n})^2(1-q^{4n-2})} \\ &= \prod_{1}^{\infty} \frac{(1-q^{10n})^4}{(1+q^{10n})^3(1+q^{2n})(1-q^{2n})^4} \\ &= \left(\prod_{n=1}^{\infty} \frac{(1-q^{10n})^2}{(1+q^{10n})(1-q^{2n})^2(1+q^{2n})^3}\right)\left(\prod_{n=1}^{\infty} \frac{(1-q^{10n})^2(1+q^{2n})^2}{(1+q^{10n})^2(1-q^{2n})^2}\right) \\ &= h(q^2)\frac{\vartheta_4^2(q^{10})}{\vartheta_4^2(q^2)}. \end{aligned}$$

This completes the proof of (18).

We also remark that if one's intention is to obtain the modular equations of degree 5, there are many to be found from the identities we have so far discussed. To mention a few, for example, the ratio of (11) and (12) yields

$$\frac{\vartheta_2^2(q) - \vartheta_2^2(q^5)}{\vartheta_3^2(q) - \vartheta_3^2(q^5)} = \sqrt{\frac{\vartheta_2(q)\vartheta_3(q^5)}{\vartheta_3(q)\vartheta_2(q^5)}}.$$

This is equivalent to

$$m = \frac{1 - \left(\frac{\beta^5}{\alpha}\right)^{\frac{1}{8}}}{1 - (\alpha^3\beta)^{\frac{1}{8}}} \qquad \left(m =: \frac{\vartheta_3^2(q)}{\vartheta_3^2(q^5)}\right).$$

From (12) and (15′), we have

$$\sqrt{\frac{\vartheta_4^5(q^5)}{\vartheta_4(q)}} - \sqrt{\frac{\vartheta_2^5(q^5)}{\vartheta_2(q)}} = \frac{1}{2}(\vartheta_3^2(q) + \vartheta_3^2(q^5))\sqrt{\frac{\vartheta_3(q^5)}{\vartheta_3(q)}}.$$

This is equivalent to

$$\left(\frac{(1-\beta)^5}{(1-\alpha)}\right)^{\frac{1}{8}} - \left(\frac{\beta^5}{\alpha}\right) = \frac{1}{2}(m+1).$$

From (15′) and (16′), we have

$$5\left\{\sqrt{\frac{\vartheta_4^5(q^5)}{\vartheta_4(q)}} - \sqrt{\frac{\vartheta_2^5(q^5)}{\vartheta_2(q)}} - \sqrt{\frac{\vartheta_3^5(q^5)}{\vartheta_3(q)}}\right\}$$
$$= \sqrt{\vartheta_2(q^5)\vartheta_2^3(q)} + \sqrt{\vartheta_3(q^5)\vartheta_3^3(q)} - \sqrt{\vartheta_4(q^5)\vartheta_4^3(q)},$$

and this, in turn, yields

$$5\left\{\left(\frac{(1-\beta)^4}{(1-\alpha)}\right)^{\frac{1}{8}} - \left(\frac{\beta^5}{\alpha}\right)^{\frac{1}{8}} - 1\right\} = m\left\{1 + (\alpha^3\beta)^{\frac{1}{8}} - ((1-\alpha)^3\beta)^{\frac{1}{8}}\right\}. \tag{20}$$

And from (19) and (20), we have

$$(\alpha^3\beta)^{\frac{1}{8}} - ((1-\alpha)^3(1-\beta))^{\frac{1}{8}} = \frac{3}{2} - \frac{5}{2m}.$$

Finally, it is worthwhile to compare our treatment with the traditional method of parametrization (see [1, pp. 280–288]) which, although less intuitive, is extremely effective in verifying the existing modular identities. It is hoped that our direct approach to this subject will make these identities less mysterious than before!

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