

## FUNCTIONS ON NONCOMPACT LIE GROUPS WITH POSITIVE FOURIER TRANSFORMS

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**ABSTRACT.** Let  $G$  be a homogeneous group with the graded Lie algebra or a noncompact semisimple Lie group with finite center. We define the Fourier transform  $\hat{f}$  of  $f$  as a family of operators  $\hat{f}(\pi) = \int_G f(x)\pi(x) dx$  ( $\pi \in \hat{G}$ ), and we say that  $\hat{f}$  is positive if all  $\hat{f}(\pi)$  are positive. Then, we construct an integrable function  $f$  on  $G$  with positive  $\hat{f}$  and the restriction of  $f$  to any ball centered at the origin of  $G$  is square-integrable, however,  $f$  is not square-integrable on  $G$ .

### 1. INTRODUCTION

When  $G$  is a compact abelian group, integrable functions  $f$  on  $G$  with the nonnegative Fourier *coefficients* and being  $p$ th ( $1 < p \leq 2$ ) power integrable near the identity of  $G$  have the Fourier coefficients in  $l^q$  ( $q = p/(p-1)$ ). This result was first obtained by N. Wiener in the case of  $G = \mathbf{T}$  and  $p = 2$  (cf. [3]) and then by Rains [9] and Ash, Rains, and Vági [1] for arbitrary compact abelian groups. When  $G$  is a compact semisimple Lie group, an analogous result was obtained by the author and Miyazaki [6]. Furthermore, Nassiet [8] and Blank [2] treated the same problem in the case that  $G$  is a compact separable group.

When  $G$  is not compact, for example, when  $G = \mathbf{R}$ , a counterexample was obtained by the author, Onoe, and Tachizawa [7]: there exists an integrable function  $f$  on  $\mathbf{R}$  with nonnegative Fourier *transform* and the restriction of  $f$  to a neighborhood of 0 is square-integrable, however,  $f$  is not square-integrable on  $\mathbf{R}$ . In this paper we shall also give a counterexample when  $G$  is a homogeneous group with the graded Lie algebra (see [4, Chapter 1]) and also when  $G$  is a noncompact semisimple Lie group with finite center. In the case of a homogeneous group with the graded Lie algebra we can find a one-parameter subgroup  $\mathcal{A}$  of  $G$  for which  $axa^{-1} = x$  for all  $a \in \mathcal{A}$  and  $x \in G$ . Then, regarding  $\mathcal{A}$  as  $\mathbf{R}$ , we can apply the same idea used in [7] to construct the counterexample on  $G$ . As compared with [7], our proof is simple and group-theoretical. Especially, the condition (3) in [7] can be replaced by a weaker condition. In the case of a noncompact semisimple Lie group we can take a one-parameter subgroup  $\mathcal{A}$  of  $G$  as a subgroup of the maximal abelian subgroup  $A$  of  $G$ . Although  $\mathcal{A}$

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does not belong to the center of  $G$ , the same idea is still applicable to obtain a counterexample.

## 2. HOMOGENEOUS GROUPS

**2.1. Notation.** Let  $G$  be a homogeneous group whose Lie algebra  $\mathfrak{g}$  is graded and  $|\cdot|: G \rightarrow \mathbf{R}_+$  a homogeneous norm of  $G$  (see [4, Chapter 1]).  $\mathfrak{g}$  is endowed with a vector space decomposition  $\mathfrak{g} = \sum_{k=1}^{\infty} V_k$  such that  $[V_i, V_j] \subset V_{i+j}$ , where all but finitely many  $V_k$ 's are  $\{0\}$ . Then if we take  $k_0 = \max\{k; V_k \neq \{0\}\}$  and  $X \neq 0 \in V_{k_0}$ ,  $\mathcal{A} = \exp(\mathbf{R}X)$  satisfies

$$(*) \quad axa^{-1} = x \quad \text{for all } a \in \mathcal{A} \text{ and } x \in G.$$

Some examples may be in order: (i) Noncompact abelian groups  $\mathbf{R}^n$ ; (ii) Heisenberg group  $H_n$ . The underlying manifold is  $\mathbf{C}^n \times \mathbf{R}$  and the multiplication law is given as

$$\begin{aligned} & (z_1, \dots, z_n, t)(z'_1, \dots, z'_n, t') \\ &= \left( z_1 + z'_1, \dots, z_n + z'_n, t + t' + 2\mathcal{I} \left( \sum_{j=1}^n z_j \bar{z}'_j \right) \right). \end{aligned}$$

Then  $\mathcal{A} = (0, \dots, 0, \mathbf{R})$  satisfies  $(*)$ ; and (iii) The group of all upper triangle matrices  $(a_{ij})_{1 \leq i, j \leq n}$  with  $a_{jj} = 1$  ( $1 \leq j \leq n$ ). Then  $\mathcal{A} = \exp(\mathbf{R}E_{1n})$  satisfies  $(*)$ , where  $E_{1n}$  is the matrix with 0 entries but 1 in the  $(1, n)$  entry. Let  $dx$  be a  $G$ -invariant measure on  $G$ . We denote the volume of a measurable set  $S$  of  $G$  by  $|S|$  and the  $L^p$ -norm ( $1 \leq p < \infty$ ) of a function  $f$  on  $G$  by  $\|f\|_p = (\int_G |f(x)|^p dx)^{1/p}$ . For any integrable functions  $f$  on  $G$  we denote the Fourier transform  $\hat{f}$  of  $f$  as a family of operators  $\hat{f}(\pi) = \int_G f(x)\pi(x) dx$  ( $\pi \in \hat{G}$ ). We say that  $\hat{f}$  is positive if all  $\hat{f}(\pi)$  ( $\pi \in \hat{G}$ ) are positive operators (see [11, p. 317]), which we denote by  $\hat{f}(\pi) \geq 0$ . Let  $B(r) = \{x \in G; |x| \leq r\}$  ( $r \in \mathbf{R}_+$ ). Then there exists a positive constant  $D$  such that

$$(**) \quad |B(r)| \sim r^D \quad (r \in \mathbf{R}_+)$$

(see [4, p. 10]). Let  $\{a_n\}_{n \in \mathbf{N}}$  be a sequence in  $\mathcal{A}$  such that  $|a_n| = n$ , and let  $\{b_n\}_{n \in \mathbf{N}}$  and  $\{r_n\}_{n \in \mathbf{N}}$  be sequences in  $\mathbf{R}_+$  satisfying

- (1)  $r_1 < 1/2$ ,  $r_n$  is decreasing, and there exists  $L \in \mathbf{R}_+$  such that  $r_n \geq 2r_m$  ( $m \geq n$ ) if and only if  $m \geq Ln$ ,
- (2)  $\sum_{n=1}^{\infty} b_n |B(r_n)| < \infty$ ,
- (3) for each  $M \in \mathbf{R}_+$ ,  $\sum_{n=1}^{\infty} \sum_{m \in \mathbf{N}, |n-m| \leq M} b_n b_m |B(r_n)|^{1/2} |B(r_m)| < \infty$ ,
- (4) for each  $N \in \mathbf{R}_+$ ,

$$\sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} \sum_{l \geq Nn'} b_n b_{n'} b_{n+l} b_{n'+l} |B(r_{n'})| |B(r_{n+l})| |B(r_{n'+l})| = \infty.$$

**Example.** Let  $(\alpha, \beta)$  be a pair of positive numbers satisfying (i)  $\alpha - \beta + 1 < 0$ , (ii)  $4\alpha - 3\beta + 2 < 0$ , and (iii)  $4\alpha - 3\beta + 3 \geq 0$ . For instance,  $\alpha = 3$  and  $\beta = 5$ . Here we let  $b_n = n^\alpha$  and  $|B(r_n)| = n^{-\beta}$ . Then (1) is obvious and (2)

follows from (i). For (3) it is enough to estimate the sum of  $n$  over  $2M$  and then

$$\sum_{n=2M}^{\infty} \sum_{\substack{m \in \mathbb{N} \\ |n-m| \leq M}} n^\alpha m^\alpha n^{-\beta/2} m^{-\beta} \leq c \sum_{n=2M}^{\infty} n^{2\alpha-3\beta/2} < \infty$$

by (ii). (4) follows from (iii) as

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} \sum_{l \geq Nn'} n^\alpha n'^\alpha (n+l)^\alpha (n'+l)^\alpha n'^{-\beta} (n+l)^{-\beta} (n'+l)^{-\beta} \\ & \geq \sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} n^\alpha n'^{\alpha-\beta} \int_{Nn'}^{\infty} (n+x)^{\alpha-\beta} (n'+x)^{\alpha-\beta} dx \\ & \geq \sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} n^\alpha n'^{3\alpha-3\beta+1} \int_N^{\infty} (x+1)^{2\alpha-2\beta} dx \\ & \geq c \sum_{n=1}^{\infty} n^\alpha \int_n^{\infty} x^{3\alpha-3\beta+1} dx \\ & \geq c \sum_{n=1}^{\infty} n^{4\alpha-3\beta+2} = \infty. \end{aligned}$$

**2.2. Counterexample.** For each measurable set  $S$  of  $G$  we denote by  $\chi_S$  the characteristic function of  $S$ . Now we define a function  $g_n$  ( $n \in \mathbb{N}$ ) on  $G$  as  $g_n(x) = b_n \mathbf{x}_n(x) = b_n \chi_{B(r_n)}(a_n^{-1}x)$  ( $x \in G$ ) and put  $g = \sum_{n=1}^{\infty} g_n$ . Then,  $\|g\|_1 = \sum_{n=1}^{\infty} \|g_n\|_1 = \sum_{n=1}^{\infty} b_n |B(r_n)| < \infty$  by (2). Here we let  $f = g^\sim * g$ , where  $g^\sim(x) = g(x^{-1})$ . Then

$$(5) \quad \|f\|_1 \leq \|g^\sim\|_1 \|g\|_1 = \|g\|_1^2 < \infty \quad \text{and} \quad \hat{f}(\pi) = \hat{g}(\pi^*) \hat{g}(\pi) \geq 0 \quad (\pi \in \hat{G}).$$

Since  $\text{supp}(\mathbf{x}_n^\sim * \mathbf{x}_m) = B(r_n) a_{m-n} B(r_m) \subset B(r_1) a_{m-n} B(r_1)$  (see (1)), it follows that  $g_n^\sim * g_m(x) = 0$  if  $x \in B(R)$  ( $R \in \mathbf{R}_+$ ) and  $|m-n| > 2r_1 + R$ . Therefore, we can deduce that for each  $R \in \mathbf{R}_+$

$$\begin{aligned} (6) \quad & \left( \int_{B(R)} |f(x)|^2 dx \right)^{1/2} \leq \sum_{n=1}^{\infty} \sum_{\substack{m \in \mathbb{N} \\ |n-m| \leq 2r_1+R}} \|g_n^\sim * g_m\|_2 \\ & \leq \sum_{n=1}^{\infty} \sum_{\substack{m \in \mathbb{N} \\ |n-m| \leq 2r_1+R}} \|g_n\|_2 \|g_m\|_1 \\ & = \sum_{n=1}^{\infty} \sum_{\substack{m \in \mathbb{N} \\ |n-m| \leq 2r_1+R}} b_n b_m |B(r_n)|^{1/2} |B(r_m)| < \infty \end{aligned}$$

by (3). Finally we show that  $\|f\|_2 = \infty$ . Since

$$\mathbf{x}_n^\sim * \mathbf{x}_m(x) = |a_n B(r_n)x \cap a_m B(r_m)| = |a_{m-n}^{-1} B(r_n)x \cap B(r_m)| \quad (x \in G),$$

it follows from (\*) that if  $m \geq n$  and  $x \in a_{m-n} B(r_n - r_m)$ , then  $a_{m-n}^{-1} B(r_n)x \supset B(r_m)$  and thus,  $\mathbf{x}_n^\sim * \mathbf{x}_m(x) = |B(r_m)|$ . On the other hand, (1) implies that there exists  $N = L - 1 \in \mathbf{R}_+$  such that  $r_n - r_m \geq r_n/2$  if and only if  $m \geq (N + 1)n$ .

Therefore, we can deduce that if  $m \geq (N + 1)n$ , then  $x_n \sim * x_m(x) = |B(r_m)|$  for  $x \in B(r_n/2)$ . Since  $|B(r)| \sim |B(2r)|$  by (\*\*), it follows from (4) that

$$\begin{aligned}
 \|f\|_2^2 &= g \sim * g * (g \sim * g) \sim (1) \\
 (7) \quad &\geq \sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} \sum_{l \geq Nn'}^{\infty} b_n b_{n'} b_{n+l} b_{n'+l} x_n \sim * x_{n+l} * x_{n'} \sim * x_{n'+l} (1) \\
 &\geq \sum_{n=1}^{\infty} \sum_{n'=n}^{\infty} \sum_{l \geq Nn'}^{\infty} b_n b_{n'} b_{n+l} b_{n'+l} |B(r_{n'})| |B(r_{n+l})| |B(r_{n'+l})| = \infty,
 \end{aligned}$$

where the symbol “ $\sim$ ” means that the ratio of the right- and left-hand sides is bounded below and above by positive constants. Then (5)–(7) implies the following

**Theorem 1.** *Let  $G$  be a homogeneous group with graded Lie algebra. Then there exists an integrable function  $f$  on  $G$  with positive  $\hat{f}$  and the restriction of  $f$  to any ball centered at the origin of  $G$  is square-integrable, however,  $f$  is not square-integrable on  $G$ .*

### 3. SEMISIMPLE LIE GROUPS

**3.1. Notation.** Let  $G$  be a noncompact semisimple Lie group with finite center and  $G = KCL(A_+)K$  a Cartan decomposition of  $G$ . Let  $\sigma: G \rightarrow \mathbf{R}_+$  denote the  $K$ -bi-invariant function on  $G$  defined by  $\sigma(x) = d(\bar{1}, \bar{x})$  ( $x \in G$ ), where  $d$  is the Riemannian distance on the symmetric space  $X = G/K$ ,  $x \rightarrow \bar{x}$  is the natural map of  $G$  to  $X$ , and  $1$  is the origin of  $G$  (cf. [10]). Let  $\mathcal{A} = \{a_t \in A; t \in \mathbf{R}\}$  be a one-parameter subgroup of  $G$  for which  $\{a_n\}_{n \in \mathbf{N}}$  is a sequence in  $A_+$  such that  $\sigma(a_n) = n$ . Let  $dx$  be a  $G$ -invariant measure on  $G$ . As in the case of homogeneous groups, we define the volume  $|S|$  of a measurable set  $S$  of  $G$ , the  $L^p$ -norm  $\|f\|_p$ , and the Fourier transform  $\hat{f}$  of a function  $f$  on  $G$ . Let  $B(r) = \{x \in G; \sigma(x) \leq r\}$ . Then there exists a positive constant  $D$  such that

$$(***) \quad |B(r)| \sim r^D \quad (r < 1)$$

(cf. [5, Chapter X]). We fix two sequences  $\{b_n\}_{n \in \mathbf{N}}$  and  $\{r_n\}_{n \in \mathbf{N}}$  in  $\mathbf{R}_+$  satisfying the exactly same conditions (1)–(4).

**3.2. Counterexample.** We define a right  $K$ -invariant function  $g_n$  ( $n \in \mathbf{N}$ ) as  $g_n(x) = b_n x_n(x) = b_n \chi_{B(r_n)}(a_n^{-1}x)$  ( $x \in G$ ). We put  $g = \sum_{n=1}^{\infty} g_n$  and define a  $K$ -bi-invariant function  $f$  on  $G$  as  $f = g \sim * g$ . By the same arguments which yield (5) and (6), we see that  $f \in L^1(G)$ ,  $\hat{f} \geq 0$ , and  $f|_{B(R)} \in L^2(G)$  for each  $R \in \mathbf{R}_+$ . Now we show that  $\|f\|_2 = \infty$ . Although (\*) does not hold for  $\mathcal{A}$ , it follows that

$$(***) \quad aB(r)a^{-1}K \supset B(r) \quad \text{for all } a \in A \text{ and } r \in \mathbf{R}_+.$$

Therefore, if  $m \geq n$  and  $x = a_{m-n}z \in a_{m-n}B(r_n - r_m)$ , we can deduce that  $a_{m-n}^{-1}B(r_n)\bar{x} \supset B(r_n)\bar{z} \supset B(r_m)$ ; and thus, applying the same argument used in the case of homogeneous groups, we can obtain that  $\|f\|_2 = \infty$ .

**Theorem 2.** *Let  $G$  be a noncompact semisimple Lie group with finite center. Then there exists an integrable  $K$ -bi-invariant function  $f$  on  $G$  with positive  $\hat{f}$  and the restriction of  $f$  to any ball centered at the origin of  $G$  is square-integrable, however,  $f$  is not square-integrable on  $G$ .*

In the proofs of Theorems 1 and 2 the structure of Lie groups is not essential. Actually, let  $G$  be a noncompact separable group and suppose that  $G$  has a one-parameter subgroup  $\mathcal{A} = \{a_t; t \in \mathbf{R}\}$  of  $G$  and the family of neighborhoods of the identity of  $G$  parametrized as  $B(r)$  ( $0 < r \leq 1$ ) satisfying (i)  $B(r) \subset B(r')$  if  $r < r'$ , (ii)  $|B(r)| \sim r^D$  for  $D > 0$ , (iii)  $a_n B(1)$  ( $n \in \mathbf{N}$ ) are disjoint, and (iv)  $aB(r)a^{-1} \supset B(r)$  for all  $a \in \mathcal{A}$  and  $0 < r \leq 1$ . Then, by the same argument used in the proof of Theorem 2, we can construct the counterexample for  $G$ .

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