PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 7, July 1995

# A SIMPLE PROOF OF A REMARKABLE CONTINUED FRACTION IDENTITY

P. G. ANDERSON, T. C. BROWN, AND P. J.-S. SHIUE

(Communicated by William W. Adams)

ABSTRACT. We give a simple proof of a generalization of the equality

$$\sum_{n=1}^{\infty} \frac{1}{2^{[n/\tau]}} = [0, 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \dots],$$

where $\tau = (1+\sqrt{5})/2$ and the exponents of the partial quotients are the Fibonacci numbers, and some closely related results.

## INTRODUCTION

P. E. Böhmer [3], L. V. Danilov [4], and W. W. Adams and J. L. Davison [1] showed independently that if $\alpha > 0$ is irrational, $b > 1$ is an integer, and $S_b(\alpha) = (b-1)\sum_{k=1}^{\infty} \frac{1}{b^{[k/\alpha]}}$, then the simple continued fraction for $S_b(\alpha)$ can be described explicitly in the following way. Let $\alpha$ have simple continued fraction

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}} = [a_0, a_1, \dots],$$

with $\frac{p_n}{q_n} = [a_0, \dots, a_n]$, $n \geq 0$. Let $t_0 = a_0 b$, $t_n = \frac{b^{q_n} - b^{q_{n-2}}}{b^{q_{n-1}} - 1}$, $n \geq 1$. Then $S_b(\alpha) = [t_0, t_1, \dots]$. Thus in the case $\alpha = \tau = (1+\sqrt{5})/2$, the golden ratio, and $b = 2$, one gets the remarkable equality $\sum_{n=1}^{\infty} \frac{1}{2^{[n/\tau]}} = [0, 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \dots]$, where the exponents of the partial quotients are the Fibonacci numbers.

More recently, R. L. Graham, D. E. Knuth, and O. Patashnik [8] indicated how to give a very different proof of the power series version of this result, where the number $b$ is replaced by an indeterminate (they carried out the proof for the case $\alpha = (1+\sqrt{5})/2$), using the continuant polynomials of Euler [6].

In this note we give a proof, which we feel is simpler than the others, which makes use of a property of the "characteristic sequence" of $\alpha$ discovered by H. J. S. Smith [13]. The crucial idea of our approach appears in Lemma 2 below, where we regard certain initial segments of the characteristic sequence of $\alpha$ as base $b$ representations of integers.

(Böhmer, Danilov, and Adams and Davison also show that $S_b(\alpha)$ is transcendental for every irrational $\alpha$. We omit the proof of this fact, which is an

Received by the editors May 25, 1993 and, in revised form, November 19, 1993.
1991 *Mathematics Subject Classification*. Primary 11A55; Secondary 11B39.

easy application of a theorem of Roth [11], using Lemma 3 and Theorem B below.)

**Preliminaries.** Let $\alpha$ be an irrational number with $0 < \alpha < 1$. (At the end, we will remove the restriction $\alpha < 1$.) Let $\alpha = [0, a_1, a_2, \dots]$ and $\frac{p_n}{q_n} = [0, a_1, \dots, a_n]$, $n \geq 0$, where $p_n, q_n$ are relatively prime non-negative integers. (As usual, we put $p_{-2} = 0$, $p_{-1} = 1$, $q_{-2} = 1$, $q_{-1} = 0$, so that $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ for all $n \geq 0$.) For $n \geq 1$, define $f_\alpha(n) = [(n+1)\alpha] - [n\alpha]$, and consider the infinite binary sequence $f_\alpha = (f_\alpha(n))_{n\geq 1}$, which is sometimes called the *characteristic sequence* of $\alpha$. Define binary words $X_n$, $n \geq 0$, by $X_0 = 0$, $X_1 = 0^{a_1-1}1$, $X_k = X_{k-1}^{a_k} X_{k-2}$, $k \geq 2$, where $X^a$ denotes the word $X$ repeated $a$ times, and $X_1 = 1$ if $a_1 = 1$.

The following result was first proved by Smith [13]. Other proofs can be found in [2], [7], [12], and [14], and further references to the characteristic sequence can be found in [2]. Nishioka, Shiokawa, and Tamura [9] treat the more general case $[(n+1)\alpha + \beta] - [n\alpha + \beta]$.

**Lemma 1.** *For each $n \geq 1$, $X_n$ is a prefix of $f_\alpha$. That is, $X_n = f_\alpha(1) f_\alpha(2) \cdots f_\alpha(s)$, where $s$ is the length of $X_n$.*

**The main proof.** We are now ready to prove the result stated in the Introduction. (However, we will keep the restriction $\alpha < 1$ until the following section.) Let $b > 1$ be an integer, let $0 < \alpha < 1$ be irrational, $\alpha = [0, a_1, a_2, \dots]$, let $\frac{p_n}{q_n} = [0, a_1, \dots, a_n]$, $n \geq 0$, and let the binary words $X_n$, $n \geq 0$, be defined as above.

According to Lemma 1, the binary word $X_n$ (which has length $q_n$ by a trivial induction using $q_n = a_n q_{n-1} + q_{n-2}$) is identical with the binary word $f_\alpha(1) f_\alpha(2) \cdots f_\alpha(q_n)$. If we let $x_n$ denote the integer whose base $b$ representation is $X_n$, i.e. $x_n = f_\alpha(1) b^{q_n - 1} + f_\alpha(2) b^{q_n - 2} + \cdots + f_\alpha(q_n) b^0$, then we can write

$$x_n = b^{q_n} \cdot \sum_{k=1}^{q_n} \frac{f_\alpha(k)}{b^k}.$$

Now we come to the crucial step.

**Lemma 2.** *For $n \geq 0$, let $t_{n+1} = \frac{b^{q_{n+1}} - b^{q_{n-1}}}{b^{q_n} - 1}$. Then for $n \geq 1$,*

$$x_{n+1} = t_{n+1} x_n + x_{n-1}.$$

*Proof.* Using the facts that $X_n$ has length $q_n$, $X_{n-1}$ has length $q_{n-1}$, $x_{n+1}$ is the integer whose base $b$ representation is $X_{n+1}$, and $X_{n+1} = X_n^{a_{n+1}} X_{n-1}$, it follows that

$$\begin{aligned} x_{n+1} &= b^{q_{n-1}}(1 + b^{q_n} + b^{2q_n} + \cdots + b^{(a_{n+1}-1)q_n}) x_n + x_{n-1} \\ &= \frac{b^{q_{n-1}}(b^{a_{n+1}q_n} - 1)}{(b^{q_n} - 1)} x_n + x_{n-1} = t_{n+1} x_n + x_{n-1}. \end{aligned}$$

**Lemma 3.** *For $n \geq 1$,*

$$[0, t_1, \dots, t_n] = \frac{b-1}{b^{q_n} - 1} \cdot x_n.$$

*Proof.* Let $y_n = \frac{b^{q_n} - 1}{b-1}$, $n \geq 0$. We show by induction on $n$ that $[0, t_1, \dots, t_n] = \frac{x_n}{y_n}$. We start the induction at $n = 0$ by setting $t_0 = 0$. Note that $x_0 = 0$,

$x_1 = 1$, $y_0 = 1$, $y_1 = \frac{b^{q_1}-1}{b-1} = t_1$. For the induction step, we simply note that $x_{n+1} = t_{n=1}x_n + x_{n-1}$ and $y_{n+1} = t_{n+1}y_n + y_{n-1}$.

**Theorem A.** *Let* $b > 1$ *be an integer, and let* $0 < \alpha < 1$ *be irrational, with* $f_\alpha(n) = [(n+1)\alpha] - [n\alpha]$, $n \geq 1$. *Let* $\alpha = [0, a_1, a_2, \ldots]$, *let* $\frac{p_n}{q_n} = [0, a_1, \ldots, a_n]$, $n \geq 0$ *(where* $p_n, q_n$ *are relatively prime non-negative integers), and let* $t_n = \frac{b^{q_n} - b^{q_{n-2}}}{b^{q_{n-1}} - 1}$, $n \geq 1$. *Then*

$$(b-1)\sum_{k=1}^{\infty} \frac{f_\alpha(k)}{b^k} = [0, t_1, t_2, \ldots].$$

*Proof.* We have seen that $x_n = b^{q_n} \sum_{k=1}^{q_n} \frac{f_\alpha(k)}{b^k}$. Hence by Lemma 3,

$$(b-1)\left(\frac{b^{q_n}}{b^{q_n}-1}\right)\sum_{k=1}^{q_n} \frac{f_\alpha(k)}{b^k} = [0, t_1, \ldots, t_n],$$

and we can take the limit as $n \to \infty$.

**Theorem B.** *With the same hypotheses as in Theorem* A, *we have*

$$(b-1)\sum_{n=1}^{\infty} \frac{1}{b^{[n/\alpha]}} = [0, t_1, t_2, \ldots].$$

*Proof.* This is a restatement of Theorem A, using the easily verified fact (when $0 < \alpha < 1$) that $f_\alpha(k) = 1$ if and only if $k = [n/\alpha]$ for some $n$.

**Theorem C.** *With the same hypotheses as in Theorem* A, *we have*

$$(b-1)^2\sum_{k=1}^{\infty} \frac{[k\alpha]}{b^k} = [0, t_1, t_2, \ldots].$$

*Proof.* Using $f_\alpha(k) = [(k+1)\alpha] - [k\alpha]$ and $[\alpha] = 0$, the series in Theorem C is obtained from the series in Theorem A by a slight rearrangement.

**Theorem D.** *With the same hypotheses as in Theorem* A, *we have*

$$\sum_{k=1}^{\infty} \frac{f_\alpha(k)}{b^k} = (b-1)\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(b^{q_k}-1)(b^{q_{k-1}}-1)}.$$

*Proof.* We saw in the proof of Lemma 3 that $[0, t_1, \ldots, t_n] = \frac{x_n}{y_n}$, $n \geq 1$, where $y_n = \frac{b^{q_n}-1}{b-1}$, $n \geq 0$. By a well-known theorem (J. B. Roberts [10, p. 101]), $\frac{x_n}{y_n} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{y_k y_{k-1}}$, $n \geq 1$, and Theorem D now follows from Theorem A.

**Removing the restriction** $\alpha < 1$. Now let $\alpha' = a_0 + \alpha$, where $a_0 \geq 0$ is an integer, $\alpha$ is irrational, and $0 < \alpha < 1$.

By Theorem A we get

$$\begin{aligned}(b-1)\sum_{k=1}^{\infty} \frac{f_{\alpha'}(k)}{b^k} &= (b-1)\sum_{k=1}^{\infty} \frac{a_0 + f_\alpha(k)}{b^k} \\ &= (b-1)a_0\sum_{k=1}^{\infty} \frac{1}{b^k} + (b-1)\sum_{k=1}^{\infty} \frac{f_\alpha(k)}{b^k} \\ &= a_0 + [0, t_1, t_2, \ldots] = [a_0, t_1, t_2, \ldots].\end{aligned}$$

To handle Theorem B we need to use the fact, whose simple proof we omit, that if $\alpha' = a_0 + \alpha$, where $0 < \alpha < 1$, then for each $k = 0, 1, 2, \ldots$, the value $k$ is assumed by the expression $[n/\alpha']$ exactly $a_0 + 1$ times if $[n/\alpha] = k$ for some $n \geq 1$, and exactly $a_0$ times if $[n/\alpha]$ never equals $k$. It then follows from Theorem B that $(b-1)\sum_{n=1}^{\infty} \frac{1}{b^{[n/\alpha']}} = [a_0 b, t_1, t_2, \ldots]$.

By Theorem C and some careful rearrangement we get $(b-1)^2 \sum_{k=1}^{\infty} \frac{[k\alpha']}{b^k} = [a_0 b, t_1, t_2, \ldots]$.

Finally, the modified Theorem D (using the modified Theorem A) is

$$(b-1)\sum_{k=1}^{\infty} \frac{f_{\alpha'}(k)}{b^k} = a_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(b-1)^2}{(b^{q_k}-1)(b^{q_{k-1}}-1)}.$$

*Remark.* This paper grew out of the first author's consideration of the number $\sum_{k=1}^{\infty} \frac{f_\alpha(k)}{2^k}$, where $\alpha = \frac{1+\sqrt{5}}{2}$, as the fixed point of the sequence $\{g_n(0)\}$, $n \geq 1$, where $g_1(x) = x/2$, $g_2(x) = (x+1)/2$, $g_n(x) = g_{n-1}(g_{n-2}(x))$, $n \geq 3$. This quickly leads (upon setting $g_n(x) = (x + a_n)/b_n$ and solving for $a_n$ and $b_n$) to

$$\sum_{k=1}^{\infty} \frac{f_\alpha(k)}{2^k} = [0, 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \ldots].$$

## Acknowledgment

The authors are grateful to the referee for references [3] and [4] and for several helpful remarks.

## References

1. W. W. Adams and J. L. Davison, *A remarkable class of continued fractions*, Proc. Amer. Math. Soc. **65** (1977), 194–198.
2. T. C. Brown, *Descriptions of the characteristic sequence of an irrational*, Canad. Math. Bull. **36** (1993), 15–21.
3. P. E. Böhmer, *Über die Transzendenz gewisser dyadischer Brüche*, Math. Ann. **96** (1926), 367–377; erratum **96** (1926), 735.
4. L. V. Danilov, *Some classes of transcendental numbers*, Math. Notes Acad. Sci. USSR **12** (1972), 524–527.
5. J. L. Davidson, *A series and its associated continued fraction*, Proc. Amer. Math. Soc. **63** (1977), 29–32.
6. L. Euler, *Specimen algorithmi singularis*, Novi Commentarii Academiae Cientiarum Petropolitanae **9** (1762), 53–69; reprinted in his Opera Omnia, Series 1, Vol. 15, pp. 31–49.
7. A. S. Fraenkel, M. Mushkin, and U. Tassa, *Determination of* $[n\theta]$ *by its sequence of differences*, Canad. Math. Bull. **21** (1978), 441–446.
8. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete mathematics*, Addison-Wesley, New York, 1989.
9. K. Nishioka, I. Shiokawa, and J. Tamura, *Arithmetical properties of a certain power series*, J. Number Theory **42** (1992), 61–87.
10. J. B. Roberts, *Elementary number theory*, MIT Press, Boston, 1977.
11. K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20; corrigendum **2** (1955), 168.
12. J. Shallit, *Characteristic words as fixed points of homomorphisms*, Univ. of Waterloo, Dept. of Computer Science, Tech. Report CS-91-72, 1991.

13. H. J. S. Smith, *Note on continued fractions*, Messenger Math. **6** (1876), 1–14.

14. K. B. Stolarsky, *Beatty sequences, continued fractions, and certain shift operators*, Canad. Math. Bull. **19** (1976), 473–482.

DEPARTMENT OF COMPUTER SCIENCE, ROCHESTER INSTITUTE OF TECHNOLOGY, ROCHESTER, NEW YORK 14623-0887

*E-mail address*: pga@cs.rit.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, SIMON FRASER UNIVERSITY, BURNABY, BRITISH COLUMBIA, CANADA V5 A1 S6

*E-mail address*: tbrown@sfu.ca

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF NEVADA, LAS VEGAS, NEVADA 89154-4020

*E-mail address*: shiue@nevada.edu