SPECTRALLY BOUNDED GENERALIZED INNER DERIVATIONS

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(Communicated by Eric Bedford)

ABSTRACT. We characterize the generalized inner derivations on a unital Banach algebra which are spectrally bounded. In particular, a simplified argument for the recent result due to Brešar that every spectrally bounded inner derivation maps into the radical is given.

In [3], a linear mapping T on a complex unital Banach algebra A was defined to be *spectrally bounded* if $r(Tx) \leq Mr(x)$ for all $x \in A$ and some $M \geq 0$, where $r(\cdot)$ denotes the spectral radius. This concept arises in connection with the non-commutative Singer-Wermer conjecture stating that the image δA of any derivation δ on A is contained in the Jacobson radical $\mathrm{rad}(A)$ if and only if $x\delta x - (\delta x)x \in \mathrm{rad}(A)$ for all $x \in A$. Although some progress on this conjecture has been made lately (see [6] and the references in [3]), it is still open and seems hard to verify.

A related, also open, conjecture is the following: *The image δA of a derivation δ on A is contained in* $\mathrm{rad}(A)$ *if and only if δ is spectrally bounded.* This was recently affirmed for *inner* derivations by Brešar [2]. His argument rests in an essential way on two results due to Pták [5], in particular on [5, Proposition 2.1] stating that a spectrally bounded inner derivation δ has the property that $\delta^2 A \subseteq \mathcal{Q}(A)$, the set of quasinilpotent elements of A. In an earlier paper, however, Pták obtained a number of equivalent conditions for the image of an inner derivation to lie in the radical. For our purposes, the following suffices.

Theorem A [4, Proposition 2.1]. *Let A be a unital Banach algebra and $a \in A$. The following conditions are equivalent.*
  (a) $ax - xa \in \mathrm{rad}(A)$ *for all $x \in A$.*
  (b) $r(ax) \leq Mr(x)$ *for all $x \in A$ and some $M \geq 0$.*

While (a) $\Rightarrow$ (b) is immediate, the implication (b) $\Rightarrow$ (a) is a consequence of the subharmonicity of the spectral radius. (But see the note at the end of this paper.)

Let L_a and R_a denote the left and right multiplications by $a \in A$, respectively. Combining Theorem A with Brešar's result we have that $L_a - R_a$ is

spectrally bounded if and only if L_a (equivalently R_a) is spectrally bounded. The main result of the present paper (Theorem B) extends this appropriately to the situation of *generalized inner derivations* $L_a - R_b$, $a, b \in A$. In fact, our proof is more direct than Brešar's argument in the case $a = b$ and uses only Theorem A, bypassing the results in [5] (which are not available for $L_a - R_b$). As a result, Brešar's theorem is an immediate consequence of Theorems A and B.

Little seems to be known about the class of spectrally bounded operators in general. Apart from the obvious ones, e.g., unital homomorphisms, there are not many examples. We therefore precede our results by compiling a few basic facts. We shall use the following additional terminology. A linear mapping $T: A \to A$ is said to be *spectrally infinitesimal* if $r(Tx) = 0$ for all $x \in A$, i.e., $TA \subseteq \mathscr{Q}(A)$, and is called *spectrally isometric* if $r(Tx) = r(x)$ for all $x \in A$, i.e, T is spectral-radius preserving.

Facts. 1. The set $SB(A)$ of all spectrally bounded operators on A is a multiplicative semigroup with identity. However, the sum of two spectrally bounded operators will in general *not* be spectrally bounded; a condition assuring this is that their images commute modulo $\mathrm{rad}(A)$.

2. The canonical epimorphism $\rho: A \to A/\mathrm{rad}(A)$ is a spectral isometry. Thus, if $T \in SB(A)$ and $S: A \to A$ is linear with $SA \subseteq \mathrm{rad}(A)$, then $T + S \in SB(A)$.

3. For each $a \in A$, L_a (equivalently, R_a) is spectrally infinitesimal if and only if $a \in \mathrm{rad}(A)$ [1, Theorem 3.1.3].

4. A spectrally infinitesimal derivation on A maps into the radical. This can be found in various places (see, e.g., [3]).

Whenever $a, b \in A$ are such that L_a and R_b are spectrally bounded, then the product $L_a R_b$ is spectrally bounded; the converse is not true in general. For the difference, however, the equivalence holds.

Theorem B. *Let A be a unital Banach algebra and $a, b \in A$. The following conditions are equivalent.*

(a) $L_a - R_b$ *is spectrally bounded.*

(b) *Both L_a and R_b are spectrally bounded.*

Proof. (a) $\Rightarrow$ (b) Note at first that it suffices to show that R_b is spectrally bounded. By Pták's result, b is then central modulo $\mathrm{rad}(A)$ wherefore

$$\begin{aligned}
r(ax - xb) &= r(\rho(a)\rho(x) - \rho(x)\rho(b)) \\
&= r((\rho(a) - \rho(b))\rho(x)) \\
&= r(\rho(x)(\rho(a) - \rho(b))) \\
&= r(\rho(b)\rho(x) - \rho(x)\rho(a)) \\
&= r(bx - xa) \quad \text{for all } x \in A.
\end{aligned}$$

Thus, $L_b - R_a$ is spectrally bounded as well, which implies that R_a, and hence L_a, is spectrally bounded.

Suppose that R_b is not spectrally bounded. Then, by Theorem A, there is an irreducible representation π of A such that $\pi(b) \notin \mathbb{C}1$. Take a vector ξ in the representation space such that $\{\xi, \pi(b)\xi\}$ is linearly independent. By Jacobson's density theorem [1, Theorem 4.2.5], we can find x and u in A

such that

$$\pi(x)\xi = 0, \quad \pi(x)\pi(b)\xi = \xi, \quad \pi(u)\pi(b)\xi = \pi(b)\xi, \quad \text{and} \quad \pi(u)\xi = -n\xi,$$

where $n \in \mathbb{N}$ is arbitrary. In fact, by Sinclair's theorem [1, Corollary 4.2.6], we may even assume that u is invertible. Then

$$(\pi(a)\pi(u)\pi(x)\pi(u)^{-1} - \pi(u)\pi(x)\pi(u)^{-1}\pi(b))\xi = n\xi,$$

whence $r(auxu^{-1} - uxu^{-1}b) \geq n$.

However, this violates the estimate

$$r(auxu^{-1} - uxu^{-1}b) \leq Mr(uxu^{-1}) = Mr(x)$$

for some constant $M \geq 0$. As a result, R_b has to be spectrally bounded.

  (b) $\Rightarrow$ (a) If L_a is spectrally bounded, so is R_a; whence, by assumption and Theorem A, R_{a-b} is spectrally bounded. In addition, $L_a - R_a$ maps into the radical. Hence, by Fact 2,

$$L_a - R_b = R_{a-b} + L_a - R_a$$

is spectrally bounded. $\quad \square$

Corollary [2, Theorem]. *Every spectrally bounded inner derivation maps into the radical.*

Proof. This is immediate from Theorems B and A. $\quad \square$

Unlike inner derivations, spectrally bounded generalized inner derivations need not map into the radical. Indeed, those that do differ from inner derivations only by an additive perturbation consisting of a multiplication by an element in the radical.

Proposition 1. *The following conditions on a, $b \in A$ are equivalent.*
  (a) $L_a - R_b$ *maps into* $\text{rad}(A)$.
  (b) $\rho(a) = \rho(b)$ *is central.*

Proof. (a) $\Rightarrow$ (b) If $(L_a - R_b)A \subseteq \text{rad}(A)$, then $a - b \in \text{rad}(A)$ and so R_{a-b} maps into $\text{rad}(A)$. Thus,

$$L_a - R_a = L_a - R_b - R_{a-b}$$

maps into $\text{rad}(A)$ as well. Consequently, $\rho(a) = \rho(b)$ is central in $A/\text{rad}(A)$. Clearly, this argument can be reversed to obtain (b) $\Rightarrow$ (a). $\quad \square$

On the other hand, Fact 4 takes over to generalized inner derivations.

Proposition 2. *Let a, $b \in A$ be such that $L_a - R_b$ is spectrally infinitesimal. Then $(L_a - R_b)A \subseteq \text{rad}(A)$.*

Proof. By Theorem B, the assumption implies that L_a is spectrally bounded; whence $L_a - R_a$ maps into $\text{rad}(A)$ by Theorem A. Moreover, the first part of the proof of Theorem B shows that $r((a-b)x) = r(ax - xb) = 0$ for all $x \in A$; hence, by Fact 3, $a - b \in \text{rad}(A)$ as well. Therefore,

$$L_a - R_b = L_a - R_a + R_{a-b}$$

maps into $\text{rad}(A)$. $\quad \square$

Note. We observe that the second part of the proof of (a) $\Rightarrow$ (b) in Theorem B shows in fact that if $(L_a - R_a)A \nsubseteq \text{rad}(A)$, i.e., a is not central modulo

rad(A), then R_a, and hence L_a, cannot be spectrally bounded. This gives a new independent argument for Theorem A, avoiding the subharmonicity of the spectral radius.

Acknowledgment

This paper was written while the second-named author enjoyed the hospitality of the University of Iowa.

Added in proof

Building on the techniques of the present paper, Brešar together with the second-named author recently affirmed the general case, i.e., *every* spectrally bounded derivation maps into the radical [*Derivations mapping into the radical, III*, J. Funct. Anal. (to appear)], whence Theorem B also extends (appropriately) to non-inner spectrally bounded generalized derivations.

References

1. B. Aupetit, *A primer on spectral theory*, Springer-Verlag, New York, 1991.
2. M. Brešar, *Derivations decreasing the spectral radius*, Arch. Math. (Basel) **61** (1993), 160–162.
3. M. Mathieu, *Where to find the image of a derivation*, Banach Center Publ. **30** (1994), 237–249.
4. V. Pták, *Derivations, commutators and the radical*, Manuscripta Math. **23** (1978), 355–362.
5. _____, *Commutators in Banach algebras*, Proc. Edinburgh Math. Soc. **22** (1979), 207–211.
6. M. Thomas, *Primitive ideals and derivations on non-commutative Banach algebras*, Pacific J. Math. **159** (1993), 139–152.

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