CRITERION FOR THE RESOLVENT SET
OF NONSYMMETRIC TRIDIAGONAL OPERATORS

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ABSTRACT. We study nonsymmetric tridiagonal operators acting in the Hilbert space ℓ^2 and describe the spectrum and the resolvent set of such operators in terms of a continued fraction related to the resolvent. In this way we establish a connection between Padé approximants and spectral properties of nonsymmetric tridiagonal operators.

1. INTRODUCTION

In this paper we consider operators in the Hilbert space ℓ^2, with the following representation in an orthonormal basis of this space:

$$(1) \qquad A = \begin{pmatrix} \beta_0 & \gamma_0 & 0 & 0 & \cdots \\ \alpha_1 & \beta_1 & \gamma_1 & 0 & \cdots \\ 0 & \alpha_2 & \beta_2 & \gamma_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

where α_k, β_k, γ_k are complex numbers, $\alpha_k \neq 0$, $\gamma_k \neq 0$, $k \in \mathbb{N}$. The operator A is defined for any finite vector $x = x_0 g_0 + x_1 g_1 + \cdots + x_n g_n$ in the orthonormal basis $\{g_n\}_0^\infty$, and its domain of definition $D(A)$ is dense in ℓ^2.

The symmetric case, where $\alpha_k = \overline{\gamma_{k-1}}$ and β_k is real, can under an appropriate chosen basis be reduced to an infinite Jacobi matrix enabling a deeper examination of this spectral theory. The main tools of investigation in this case are the classical moment problem (including the theory of selfadjoint extensions of unbounded symmetric operators), the theory of general orthogonal polynomials and the spectral theorem for selfadjoint operators ([1], [2], [9], [11]). This connection between the spectral theory and analysis is fruitful for various points of view. For example, the scattering problem for a Jacobi matrix can be treated on the basis of strong (or Szegő type) asymptotic results for orthogonal polynomials ([4], [6], [10]). On the other hand, the perturbation theory gives new results for orthogonal polynomials ([12]).

In the general, nonsymmetric, case we cannot use the spectral theorem and orthogonal polynomials, but in return we will obtain results of use for Padé approximants and the theory of continued fractions. We denote as usual by $\sigma(A)$ the spectrum of the operator A, $\Omega(A) = \mathbb{C} \setminus \sigma(A)$ is the resolvent set and the resolvent is $R(\lambda) = (\lambda I - A)^{-1}$ for $\lambda \in \Omega$. By $\{g_n\}_0^\infty$ we denote the orthonormal basis in ℓ^2. The function

$$(2) \qquad \phi(\lambda) = (R(\lambda)g_0, g_0)$$

is analytic on the resolvent set $\Omega(A)$. If the operator A is bounded (this is the case when α_k, β_k, γ_k are bounded), then the function $\phi(\lambda)$ is analytic for $|\lambda| > \|A\|$. For convenience we have changed the basis of representation of the operator A. The new basis $\{e_n\}_0^\infty$ is defined by $e_n = g_n/d_n$, where $d_n = \gamma_0 \gamma_1 \ldots \gamma_{n-1}$, $d_0 = 1$. In this basis the operator (1) has the following form:

$$(3) \qquad A = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & \ldots \\ \alpha_1\gamma_0 & \beta_1 & 1 & 0 & \ldots \\ 0 & \alpha_2\gamma_1 & \beta_2 & 1 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

If we set $a_n = \alpha_n\gamma_{n-1}$, $b_n = \beta_n$, then we have in the basis $\{e_n\}$:

$$(4) \qquad A = \begin{pmatrix} b_0 & 1 & 0 & 0 & \ldots \\ a_1 & b_1 & 1 & 0 & \ldots \\ 0 & a_2 & b_2 & 1 & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The Chebyshev algorithm (Algorithm 7.2.1 in [8, page 248]) applied to $\phi(\lambda)$ at infinity gives us the following continued fraction (J-fraction):

$$(5) \qquad \frac{1|}{|\lambda - b_0} - \frac{a_1|}{|\lambda - b_1} - \frac{a_2|}{|\lambda - b_2} - \cdots .$$

The numerators P_n and denominators Q_n of the nth convergents for this fraction satisfy the three-term recurrence relation:

$$(6) \qquad a_n y_{n-1} + b_n y_n + y_{n+1} = \lambda y_n, \qquad n = 0, 1, 2, \ldots ,$$

with the initial conditions

$$\begin{cases} Q_{-1} = 0, & Q_0 = 1, \\ P_{-1} = 1, & P_0 = 0. \end{cases}$$

The main problem considered in this paper is to describe the spectrum and the resolvent set of operators of type (1) in terms of the continued fraction (5) and the monic polynomials P_n, Q_n and in this way to establish a connection between Padé approximants and spectral properties of nonsymmetric tridiagonal operators. In our main result, Theorem 1, we state the criterion for the resolvent set $\Omega(A)$ in terms of the growth of the polynomials $Q_n(z)$. The proof of the necessary condition is essentially based on the results of Kershaw and Demko, Moss, and Smith about decay rates of inverses of banded matrices ([5]). This theorem gives us analytic properties of the polynomials $Q_n(\lambda)$ and the related Padé approximants $\pi_n := P_n/Q_n$ on the set $\Omega(A)$. We prove, among other things, the following

Theorem 2. *If A is bounded and*

$$0 < |\alpha_k|, |\gamma_k| \leq C_1 < \infty$$

for some C_1, then for any $\lambda \in \Omega(A)$ there is a subsequence of Padé approximants π_n which converges to $\phi(\lambda)$ with a geometric rate.

In section 2 we formulate and prove Theorem 1 and related results. Section 3 will be devoted to an application of these results to the behavior of Padé approximants.

2. ANALYSIS OF THE SPECTRUM AND THE RESOLVENT SET OF THE OPERATOR

Suppose A has a representation (4) in an orthogonal basis $\{e_n\}$, where $\|e_n\| = h_n$. We assume that the operator A is bounded. In this case A is defined on the whole space ℓ^2 and $\lambda \in \Omega(A)$ if and only if the following conditions are satisfied:

(1) $\text{Ker}(\lambda I - A) = \{0\}$,
(2) $\forall n, \exists z_n : (\lambda I - A)z_n = e_n$,
(3) $\|(\lambda I - A)^{-1}x\| \leq C\|x\|$, for all finite vectors $x = x_0 e_0 + x_1 e_1 + \cdots + x_n e_n$ and some constant C.

Proposition 1. *λ_0 is an eigenvalue of A if and only if*

$$(7) \qquad \sum_{n=0}^{\infty} |Q_n(\lambda_0)|^2 h_n^2 < +\infty.$$

Proof. We can formally write $Ax = \lambda_0 x$ and obtain for $x = x_0 e_0 + x_1 e_1 + \cdots$ the following system:

$$\begin{cases} b_0 x_0 + x_1 = \lambda_0 x_0 \\ a_1 x_0 + b_1 x_1 + x_2 = \lambda_0 x_1 \\ a_2 x_1 + b_2 x_2 + x_3 = \lambda_0 x_2 \\ \qquad \vdots \qquad \qquad \vdots \quad \vdots \end{cases}$$

If $x_0 = 0$, then $x_n = 0$, $\forall n \in \mathbb{N}$. If $x_0 \neq 0$, then $x_n = Q_n(\lambda_0)x_0$ and in this case $x \in \ell^2$ iff $\sum_0^{\infty} |Q_n(\lambda_0)|^2 h_n^2 < \infty$. $\square$

Proposition 2. *For given λ, the equation*

$$(\lambda I - A)z = e_n$$

has a solution in ℓ^2 for all $n \geq 0$ if and only if there exists a complex number γ such that

$$(8) \qquad \sum_{n=0}^{\infty} |Q_n(\lambda)\gamma - P_n(\lambda)|^2 h_n^2 < +\infty.$$

Proof. First we find the formal solution $z^{(0)}$ of the equation

$$(\lambda I - A)z = e_0.$$

Let $z^{(0)} = (z_{0,0}, z_{1,0}, z_{2,0}, \dots)$ in the basis $\{e_n\}$; then

$$(9) \qquad \begin{cases} 1 + b_0 z_{0,0} + z_{1,0} = \lambda z_{0,0} \\ a_1 z_{0,0} + b_1 z_{1,0} + z_{2,0} = \lambda z_{1,0} \\ a_2 z_{1,0} + b_2 z_{2,0} + z_{3,0} = \lambda z_{2,0} \\ \qquad \vdots \qquad \qquad \vdots \quad \vdots \end{cases}$$

So $z_{n,0}$ satisfies the same recurrence relations as $Q_n(\lambda)$ and $P_n(\lambda)$ for $n \geq 1$, only the first relation is different. This implies that $z_{n,0} = Q_n(\lambda)\gamma - P_n(\lambda)$ for $n > 1$ and for any γ the first relation is also satisfied: $b_0\gamma + \gamma(\lambda - b_0) - 1 + 1 = \lambda\gamma$. Thus the sequence $Q_n(\lambda)\gamma - P_n(\lambda)$ is a solution of (9) for any γ. We need a solution in ℓ^2. So e_0 is in the image of $(\lambda I - A)$ iff $\sum_0^\infty |Q_n(\lambda)\gamma - P_n(\lambda)|^2 h_n^2 < +\infty$ for some γ. Note that if λ is not an eigenvalue of A, then the convergence of this series is possible only for one γ. For the solution of the equation $(\lambda I - A)z = e_1$ we note that $(\lambda I - A)e_0 = \lambda e_0 - Ae_0 = (\lambda - b_0)e_0 - a_1 e_1$ and then $-a_1 e_1 = (\lambda I - A)(e_0 - (\lambda - b_0)z^{(0)})$. So $z^{(1)} = -(e_0 - (\lambda - b_0)z^{(0)})/a_1$ is a solution of $(\lambda I - A)z = e_1$. This is an element of ℓ^2 if $z^{(0)} \in \ell^2$. In the same way we find $z^{(2)} = -(e_1 - (\lambda - b_1)z^{(1)} + z^{(0)})/a_2$ for the solution of $(\lambda I - A)z = e_2$ and so on. $\square$

Theorem 1. *Suppose A is bounded, λ is not an eigenvalue of A, and (8) is satisfied for some γ. Then $\lambda \in \Omega(A)$ if and only if there are constants $C > 0$ and $0 < q < 1$ such that*

$$(10) \qquad \left| \frac{Q_n(\lambda)r_m(\lambda)}{a_1 a_2 \ldots a_n} \frac{h_m}{h_n} \right| \leq Cq^{m-n}, \qquad n \leq m,$$

$$(11) \qquad \left| \frac{Q_m(\lambda)r_n(\lambda)}{a_1 a_2 \ldots a_n} \frac{h_m}{h_n} \right| \leq Cq^{n-m}, \qquad n \geq m,$$

where $r_m(\lambda) := Q_m(\lambda)\gamma - P_m(\lambda)$.

Proof. From Propositions 1 and 2 it follows that the inverse operator $B = (\lambda I - A)^{-1}$ can be defined on the basis vectors by $Be_n = z^{(n)}$. The matrix of B in the basis $\{e_n\}$ is of the form:

$$(12) \qquad \begin{pmatrix} r_0 & \frac{r_1}{a_1} & \frac{r_2}{a_1 a_2} & \frac{r_3}{a_1 a_2 a_3} & \cdots \\ r_1 & Q_1 \frac{r_1}{a_1} & Q_1 \frac{r_2}{a_1 a_2} & Q_1 \frac{r_3}{a_1 a_2 a_3} & \cdots \\ r_2 & r_2 \frac{Q_1}{a_1} & Q_2 \frac{r_2}{a_1 a_2} & Q_2 \frac{r_3}{a_1 a_2 a_3} & \cdots \\ r_3 & r_3 \frac{Q_1}{a_1} & r_3 \frac{Q_2}{a_1 a_2} & Q_3 \frac{r_3}{a_1 a_2 a_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Indeed, for the first column there is no problem, since it contains $z^{(0)}$. The nth column contains $z^{(n)}$. For $z^{(1)} = -(e_0 - (\lambda - b_0)z^{(0)})/a_1$ we have

$$(13) \qquad \begin{aligned} z_{0,1} &= -\frac{1 - (\lambda - b_0)r_0}{a_1} = \frac{r_1}{a_1} \\ z_{1,1} &= -\frac{-(\lambda - b_0)z_{1,0}}{a_1} = Q_1 \frac{r_1}{a_1} \\ z_{2,1} &= -\frac{-(\lambda - b_0)z_{2,0}}{a_1} = r_2 \frac{Q_1}{a_1} \end{aligned}$$

$$\vdots \quad \vdots \qquad \vdots$$

This is the second column of B. Suppose we have calculated $z^{(k)}$ as indicated and we have to find $z^{(k+1)}$. First note that

$$z^{(k+1)} = \frac{e_k - (\lambda - b_k)z^{(k)} + z^{(k-1)}}{a_{k+1}}.$$

Then

$$z_{0,k+1} = -\frac{1}{a_{k+1}}(-(\lambda - b_k)\frac{r_k}{a_1 a_2 \cdots a_k} + \frac{r_{k-1}}{a_1 a_2 \cdots a_{k-1}}) = \frac{r_{k+1}}{a_1 a_2 \cdots a_{k+1}}$$

$$z_{1,k+1} = -\frac{1}{a_{k+1}}(-(\lambda - b_k)\frac{Q_1 r_k}{a_1 a_2 \cdots a_k} + \frac{Q_1 r_{k-1}}{a_1 a_2 \cdots a_{k-1}}) = Q_1 \frac{r_{k+1}}{a_1 a_2 \cdots a_{k+1}}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$z_{k,k+1} = -\frac{1}{a_{k+1}}(1 - (\lambda - b_k)\frac{Q_k r_k}{a_1 a_2 \cdots a_k} + \frac{Q_{k-1} r_k}{a_1 a_2 \cdots a_{k-1}}) = Q_k \frac{r_{k+1}}{a_1 a_2 \cdots a_{k+1}}.$$

We used here the relations $Q_{k-1} r_k - Q_k r_{k-1} = -a_1 a_2 \cdots a_{k-1}$ deduced from the recurrence relations for Q_k and r_k. Next,

$$z_{k+1,k+1} = -\frac{1}{a_{k+1}}(-(\lambda - b_k)\frac{Q_k r_{k+1}}{a_1 a_2 \cdots a_k} + \frac{Q_{k-1} r_{k+1}}{a_1 a_2 \cdots a_{k-1}}) = r_{k+1} \frac{Q_{k+1}}{a_1 a_2 \cdots a_{k+1}}$$

and so on for $z_{l,k+1}$, $l > k + 1$. Thus the form of the matrix B is correct. Now we decompose B into two parts: $B = B_1 + B_2$, where B_1 and B_2 are lower and upper triangular matrices with the same diagonal elements $(B_1)_{i,i} = (B_2)_{i,i} = \frac{1}{2}B_{i,i}$. For the estimation of norms of matrices we use the formula

$$\|K\| = \sup_{\|x\| \leq 1} \sup_{\|y\| \leq 1} |(y, Kx)|.$$

We have for two finite vectors $x = x_0 e_0 + x_1 e_1 + \cdots + x_n e_n$ and $y = y_0 e_0 + y_1 e_1 + \cdots + y_n e_n$:

$$(y, B_1 x) = \frac{1}{2}\overline{b_{0,0}} y_0 \overline{x_0} h_0^2 + (\overline{b_{1,0}} y_1 \overline{x_0} + \frac{1}{2}\overline{b_{1,1}} y_1 \overline{x_1}) h_1^2$$

$$+ \cdots + (\overline{b_{n,0}} y_n \overline{x_0} + \overline{b_{n,1}} y_n \overline{x_1} + \cdots + \frac{1}{2}\overline{b_{n,1}} y_n \overline{x_n}) h_n^2.$$

Here, we denote by $b_{i,j}$ the elements of the matrix B. Then

$$|(y, B_1 x)| \leq \frac{1}{2}\sum_{k=0}^{n} |b_{k,k}||x_k||y_k| h_k^2 + \sum_{j=1}^{n}\sum_{k=0}^{n-j} |b_{k+j,k}||x_k||y_{k+j}| h_{k+j}^2$$

$$= \frac{1}{2}\sum_{k=0}^{n} |b_{k,k}||x_k||y_k| h_k^2 + \sum_{j=1}^{n}\sum_{k=0}^{n-j} |b_{k+j,k}| \frac{h_{k+j}}{h_k} ||x_k h_k||y_{k+j} h_{k+j}|$$

$$\leq C(\frac{1}{2} + q + q^2 + \cdots + q^n)\|x\| \|y\|$$

$$\leq C\frac{1 - q^{n+1}}{1 - q}\|x\| \|y\|.$$

A similar estimation is applicable for the matrix B_2. The first part of the theorem is thus proved.

To prove that for $\lambda \in \Omega(A)$ the estimation (10) holds, one can use the known theorem on the decay rates of the inverse of banded matrices (see [5]) and the matrix representation of $B = (\lambda I - A)^{-1}$ obtained in the first part of our proof. $\square$

3. APPLICATION TO PADÉ APPROXIMANTS

In this section we consider some applications of Theorem 1. First of all we note that Theorem 1 together with Propositions 1 and 2 gives a characterization of the resolvent set of A in terms of the polynomials P_n and Q_n. On the other hand this theorem also gives a result on the convergence of the continued fractions (5). We recall that the fraction P_n/Q_n is a diagonal type Padé approximant for $\phi(\lambda)$ at infinity (see, e.g., Theorem 7.15 (B) in [8, page 250]).

Corollary 2. *If $\lambda \in \Omega(A)$, then the remainder of Padé approximation in linear form tends to zero, i.e.,*

$$\lim_{n \to \infty} [Q_n(\lambda)\phi(\lambda) - P_n(\lambda)]h_n = 0.$$

Proof. We have $z^{(0)} = R(\lambda)e_0$ and $\phi(\lambda) = (R(\lambda)e_0, e_0) = z_{0,0}$, with $h_0 = 1$. On the other hand $z_{0,0} = \gamma Q_0 - P_0 = \gamma$; hence $\phi(\lambda) = \gamma$ and the corollary follows from Proposition 2. □

Remark 1. In the case of a symmetric operator A with a representation (1) in some orthogonal basis, the polynomials

$$q_n = Q_n/d_n, \quad d_n = \gamma_0\gamma_1 \cdots \gamma_{n-1},$$

are orthonormal and if $\lambda \in \Omega(A)$, then $q_n(\lambda)\phi(\lambda) - p_n(\lambda) \to 0$ as $n \to \infty$. In this case we have $h_n = 1/|d_n|$.

Corollary 3. *If there exists a positive constant C_1 such that $|\alpha_k|, |\gamma_k| \le C_1$, then for all $\lambda \in \Omega(A)$*

$$\limsup_{n \to \infty} |Q_n(\lambda)h_n|^{1/n} > 1.$$

Proof. Both Q_n and r_n satisfy the same recurrence relation (6). Hence

$$Q_{n-1}r_n - Q_nr_{n-1} = -a_1a_2 \cdots a_{n-1},$$

or equivalently

$$Q_{n-1}h_{n-1}\frac{r_n}{h_na_1 \cdots a_n}a_n\frac{h_n}{h_{n-1}} - Q_nh_n\frac{r_{n-1}}{h_{n-1}a_1 \cdots a_{n-1}}\frac{h_{n-1}}{h_n} = -1.$$

From Theorem 1 we get

$$\left|\frac{r_k}{h_ka_1 \cdots a_k}\right| \le Cq^k, \quad q < 1,$$

so the sequence Q_nh_n cannot be majorized by a geometric sequence p^n with $p < 1/q$, i.e., for any $p < 1/q$ and any positive constant C the inequality

$$|Q_n(\lambda)h_n| \le Cp^n$$

is not satisfied for an infinite number of indices n. For a subsequence $\Lambda \subset \mathbb{N}$ we thus have

$$|Q_n(\lambda)h_n| \ge Cp^n, \quad n \in \Lambda;$$

and consequently if we choose p such that $1/q > p > 1$ we have

$$\limsup_{n \to \infty} |Q_n(\lambda)h_n|^{1/n} \ge p > 1,$$

giving the required result. □

Remark 2. In the symmetric case this corollary gives us the well-known characterization of $\Omega(A)$ in terms of the orthonormal polynomials q_n: if $\lambda \in \Omega(A)$, then $\limsup_{n \to \infty} |q_n(\lambda)|^{1/n} > 1$ ([11]).

The combination of these two corollaries gives

Theorem 2. *If A is bounded and $0 < |\alpha_k|, |\gamma_k| \leq C_1 < \infty$ for some constant C_1, then for any $\lambda \in \Omega(A)$ there exists a subsequence of the Padé approximants $\pi_n = P_n(\lambda)/Q_n(\lambda)$ which converges with a geometric rate to $\phi(\lambda)$.*

Proof. From Corollary 1 we have

$$\lim_{n \to \infty} Q_n h_n \phi(\lambda) - P_n h_n = 0.$$

Let $\Lambda \subset \mathbb{N}$ be such that

$$\lim_{n \to \infty} |Q_n(\lambda) h_n|^{1/n} > 1,$$

then $|Q_n(\lambda) h_n| \geq C p^n$ for some $p > 1$. Hence

$$\left| \phi(\lambda) - \frac{P_n(\lambda)}{Q_n(\lambda)} \right| \leq C \left(\frac{q}{p} \right)^n,$$

which proves the theorem. $\square$

Remark 3. In the symmetric case this statement seems to be unnoticed. Indeed, suppose A is in the form (1) in an orthonormal basis, and consider the so-called asymptotically periodic case (or the corresponding limit-periodic continued fraction) where for some N we have

$$\lim_{k \to \infty} \alpha_{kN+i} = \delta_i, \qquad i = 0, 1, 2, \ldots N - 1,$$

and similarly for β_{kN+i} and γ_{kN+i}. Then it is known that the poles of the Padé approximants $\pi_n = P_n/Q_n$ are essentially concentrated on a system of N closed intervals ([4], [7]). The convergence of π_n however depends on the behaviour of the so-called spurious poles. Nevertheless our theorem shows that for any λ in the exterior of the spectrum some subsequence of π_n converges with a geometric rate.

Finally we note that from Theorem 1 we get a connection between the fact that $\lambda \in \Omega(A)$ and the convergence of Padé approximants $\pi_n(\lambda) = P_n(\lambda)/Q_n(\lambda)$. The convergence of $\pi_n(\lambda)$, without further requirements on the rate, does not imply that $\lambda \in \Omega(A)$. A counterexample is given by the spectral measure $d\mu(x) = \sqrt{1 - x^2}\, dx$ on $[-1, 1]$, for which $P_n(x) = 2^{-n+1} U_{n-1}(x)$ and $Q_n(x) = 2^{-n} U_n(x)$, where $U_n(x)$ are the Chebyshev polynomials of the second kind. At $\lambda = 1 \in \sigma(A)$ we have $\pi_n(1) = 2n/(n+1)$ which converges, but without geometric rate. The convergence of some subsequence with a geometric rate is also not sufficient to imply that $\lambda \in \Omega(A)$. A counterexample is given by any symmetric measure on $[-1, 1]$, because then at $\lambda = 0 \in \sigma(A)$ we always have $P_{2n}(\lambda) = 0$.

The following question is thus of interest: does the convergence of the whole sequence $\pi_n(\lambda)$ with a geometric rate imply that $\lambda \in \Omega(A)$?

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