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TRANSFERENCE OF MAXIMAL MULTIPLIERS ON HARDY SPACES

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ABSTRACT. Based on the atomic decomposition of the Hardy space, we give a simple proof for a theorem of Liu and Lu (Studia Math. **105** (1993), 121–134), which discusses the relation between the maximal operators on $\mathbb{R}^n$ and on $\mathbb{T}^n$. More significantly, our proof shows that condition (1) in Liu and Lu's Theorem 1 is superfluous.

1. INTRODUCTION

Let $H^p(\mathbb{R}^n)$, $0 < p < \infty$, be the Hardy spaces defined by [FS]

$$H^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n),\ \|\Phi^+ f\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where $\Phi^+ f(x) = \sup_{t>0} |\Phi_t * f(x)|$, $\Phi_t(x) = t^{-n}\Phi(x/t)$, and $\Phi \in \mathcal{S}(\mathbb{R}^n)$ is a radial function satisfying $\int \Phi = 1$. The corresponding periodic Hardy spaces are $H^p(\mathbb{T}^n) = \{f \in \mathcal{S}'(\mathbb{T}^n),\ \|\tilde{\Phi}^+ f\|_{L^p(\mathbb{T}^n)} < \infty\}$, where $\Phi^+ f(x) = \sup_{t>0} |\tilde{\Phi}_t * f(x)|$, $\tilde{\Phi}_t(x) = \sum_{k\in\Lambda} \hat{\Phi}(tk) e^{2\pi i k\cdot x} = C t^{-n} \sum_{k\in\Lambda} \Phi((x+k)/t)$ and Λ is the unit lattice which is the additive group of points in $\mathbb{R}^n$ having integral coordinates.

Let λ be a bounded continuous function on $\mathbb{R}^n$. For each $\varepsilon > 0$, define

$$(T_\varepsilon f)\hat{}\,(u) = \lambda(\varepsilon u)\hat{f}(u), \qquad f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n),$$

and

$$\widetilde{T}_\varepsilon f(x) = \sum_{k\in\Lambda} \lambda(\varepsilon k) a_k(f) e^{2\pi i k\cdot x}, \qquad f \in L^2(\mathbb{T}^n) \cap H^p(\mathbb{T}^n).$$

We say that λ is a maximal multiplier on $H^p(\mathbb{R}^n)$ if $T^* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$ can be extended to a bounded operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Similarly, λ is called a maximal multiplier on $H^p(\mathbb{T}^n)$ if $\widetilde{T}^* f(x) = \sup_{\varepsilon>0} |\widetilde{T}_\varepsilon f(x)|$ can be extended to a bounded operator from $H^p(\mathbb{T}^n)$ to $L^p(\mathbb{T}^n)$.

The relation between the maximal multipliers T^* and $\widetilde{T}^*$ was first studied by Kenig and Tomas when $p > 1$. Their result can be extended to the Lorentz

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space $L(p, q)$, $p > 1$ (see [F]). Recently, Liu and Lu [LL] studied the case of $0 < p \leq 1$. Their main result is the following theorem.

Theorem 1 [LL]. *Let $0 < p \leq 1$, and let λ be a bounded and continuous function on $\mathbb{R}^n$.*

(i) *Suppose that λ is a maximal multiplier on $H^p(\mathbb{R}^n)$ such that*

$$\lim_{|x|\to\infty} \lambda(x) = \alpha \tag{1}$$

exists. Then λ is a maximal multiplier on $H^p(\mathbb{T}^n)$.

(ii) *If λ is a maximal multiplier on $H^p(\mathbb{T}^n)$, then λ is a maximal multiplier on $H^p(\mathbb{R}^n)$.*

Since the above condition (1) plays an important role in their proof, Liu and Lu asked (see p. 133 in [LL]) if condition (1) can be weakened any further.

In this note, we will show that condition (1) in Theorem 1 is superfluous. Based on the atomic decomposition of the Hardy space, our proof is much shorter and more direct than those in [LL]. The following is our main result.

Theorem 2. *Let λ be a continuous and bounded function on $\mathbb{R}^n$, $0 < p \leq 1$. If*

$$\|T^* f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)} \quad \textit{for all } f \in H^p(\mathbb{R}^n),$$

then

$$\|\widetilde{T}^* \tilde{f}\|_{L^p(\mathbb{T}^n)} \leq C\|\tilde{f}\|_{H^p(\mathbb{T}^n)} \quad \textit{for all } \tilde{f} \in H^p(\mathbb{T}^n).$$

For the sake of simplicity, C always denotes a positive constant which may vary at each of its occurrences.

To prove Theorem 2, we need to use the atomic characterization of the Hardy space. A regular $(p, 2, s)$ atom is a function $a(x)$ supported in some ball $B(x_0, \rho)$ satisfying

$$\|a\|_2 \leq \rho^{-n/p+n/2}; \tag{i}$$

$$\int_{\mathbb{R}^n} a(x)P(x)\, dx = 0 \tag{ii}$$

for all polynomials $P(x)$ of degree less than or equal to s.

The space $H_a^{p,s}(\mathbb{R}^n)$, $0 < p \leq 1$, is the space of all distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ having the form

$$f = \sum c_k a_k \tag{2}$$

and satisfying

$$\sum |c_k|^p < \infty, \tag{3}$$

where each a_k is a $(p, 2, s)$ atom. The "norm" $\|f\|_{H_a^{p,s}(\mathbb{R}^n)}$ is the infimum of all expressions $(\sum |c_k|^p)^{1/p}$ for which we have a representation (2) of f. A well-known fact (see [FoS]) is that $\|f\|_{H_a^{p,s}(\mathbb{R}^n)} \cong \|f\|_{H^p(\mathbb{R}^n)}$, and in particular, $\|a\|_{H^p(\mathbb{R}^n)} \leq C$, with a constant C independent of the $(p, 2, s)$ atom $a(x)$ if $s \geq [n(1/p - 1)]$.

We also have a similar decomposition theorem for any function $g \in H^p(\mathbb{T}^n)$. In particular, suppose $g \in H^p(\mathbb{T}^n) \cap \mathscr{S}(\mathbb{T}^n)$ and its Fourier coefficient

$$a_0(g) = \int_Q g(x)\, dx = 0,$$

where $Q = \{x \in \mathbb{R}^n : -1/2 \le x_j < 1/2, j = 1, 2, \dots, n\}$ is the fundamental cube on which

$$\int_{\mathbb{T}^n} g(x)\,dx = \int_Q g(x)\,dx$$

for all functions g on $\mathbb{T}^n$. Then we have the following lemma.

Lemma 4. *Suppose $g \in H^p(\mathbb{T}^n) \cap \mathscr{S}(\mathbb{T}^n)$ with $a_0(g) = 0$. If we restrict x to Q, then for any fixed positive integer s*

$$g(x) = \sum c_k a_k(x),$$

where each $a_k(x)$ is a $(p, 2, s)$ atom satisfying $a_k(x+n) = a_k(x)$ for $n \in \Lambda$ and $\|g\|^p_{H^p(\mathbb{T}^n)} \cong \sum |c_k|^p$.

Proof. Choose a radial function $\varphi \in \mathscr{S}(\mathbb{R}^n)$ with $\operatorname{supp}(\varphi) \subset B(0, 1)$. In addition, we can choose such a φ such that $\int x^J \varphi(x)\,dx = 0$ for all multi-indices J, $|J| \le s$, and $\int_0^\infty \hat{\varphi}(tx)^2 t^{-1} dt = 1$ for all $x \ne 0$. We let $\tilde{\varphi}_t(x) = \sum_{k \in \Lambda \setminus \{0\}} \hat{\varphi}(tk) e^{2\pi i k \cdot x} = C \sum_{k \in \Lambda} t^{-n} \varphi((x+k)/t)$. Then by checking the Fourier coefficients, we easily obtain the following Calderón reproducing formula:

$$g(x) = \int_0^\infty (\tilde{\varphi}_t * \tilde{\varphi}_t * g)(x) t^{-1} dt = \int_0^1 + \int_1^\infty. \tag{5}$$

Now by a standard argument [FoS] (or see [BF] for the proof on any compact Lie group), one can easily obtain that

$$g(x) = \sum c_k a_k(x),$$

where each $a_k(x)$ is a $(p, 2, s)$ atom satisfying $a_k(x+n) = a_k(x)$ for $n \in \Lambda$ and $\sum |c_k|^p \cong \|S_\varphi(g)\|^p_{L^p(\mathbb{T}^n)}$. Here $S_\varphi(g)$ is defined by

$$S_\varphi(g)(x) = \int_{|x-y|<t} |(g * \tilde{\varphi}_t)(y)|^2 t^{-n-1}\,dy\,dt.$$

So to prove the lemma it suffices to show that $\|S_\varphi(g)\|_{L^p(\mathbb{T}^n)} \cong \|g\|_{H^p(\mathbb{T}^n)}$ for all $g \in H^p(\mathbb{T}^n) \cap \mathscr{S}(\mathbb{T}^n)$. But the proof for $\|S_\varphi(g)\|_{L^p(\mathbb{T}^n)} \cong \|g\|_{H^p(\mathbb{T}^n)}$ is, mutatis mutandis, the same as for $\mathbb{R}^n$ (see [FoS]) without using any new techniques or ideas.

The following lemma is Lemma 3.1 in [F]. For completeness, we state its proof.

Lemma 6. *Suppose that $\Psi(x)$ is a continuous function with compact support. Let $\lambda(x)$ be a bounded and continuous function on $\mathbb{R}^n$, and let T_ε and $\widetilde{T}_\varepsilon$ be the families of operators on $\mathbb{R}^n$ and $\mathbb{T}^n$, respectively, associated to the function λ. Take $\Psi^{1/N}(\xi) = \Psi(\xi/N)$. If Ψ satisfies $\Psi(0) = 1$ and $\hat{\Psi} \in L^1(\mathbb{R}^n)$, then for any $g \in \mathscr{S}(\mathbb{T}^n)$ and any positive integer N,*

$$\Psi(y/N)(\widetilde{T}_\varepsilon g)(y) = T_\varepsilon(g\Psi^{1/N})(y) + J_{N,\varepsilon}(y) \tag{7}$$

for all $y \in \mathbb{R}^n$, where $J_{N,\varepsilon}(y)$ tends to zero uniformly for $y \in \mathbb{R}^n$ and $0 \le \varepsilon \le R$ ($R > 0$ is any fixed number), as $N \to \infty$.

Proof. Since $g(x) = \sum a_k(g) e^{2\pi i k \cdot x}$ with the Fourier coefficients $\{a_k(g)\}$ rapidly decreasing as $|k| \to \infty$, it suffices to prove the lemma when $g(x) =$

$e_k(x) = e^{2\pi i k\cdot x}$. In this case

$$\begin{aligned}|J_{N,\varepsilon}(y)| &= |\Psi^{1/N}(y)(\widetilde{T}_\varepsilon e_k)(y) - T_\varepsilon(\Psi^{1/N}e_k)(y)| \\ &= \left| e_k(y)\int_{\mathbb{R}^n} N^n\hat{\Psi}(N\xi)e^{2\pi i y\cdot\xi}\{\lambda(\varepsilon k) - \lambda(\varepsilon k + \varepsilon\xi)\}\,d\xi\right| \\ &\le \int_{\mathbb{R}^n} |\widehat{\Psi}(\xi)|\,|\lambda(\varepsilon k) - \lambda(\varepsilon k + N^{-1}\varepsilon\xi)|\,d\xi.\end{aligned}$$

Since $\hat{\psi}$ is integrable and since λ is bounded and continuous, the last quantity converges to zero as $N\to\infty$. The lemma is proved.

2. Proof of the main theorem

By a note on page 128 in [LL] we only need to show that for any $g \in \mathscr{S}(\mathbb{T}^n)\cap H^p(\mathbb{T}^n)$ with $a_0(g) = 0$,

$$\|\widetilde{T}^* g\|_{L^p(\mathbb{T}^n)} \le C\|g\|_{H^p(\mathbb{T}^n)}. \tag{8}$$

For any $R>0$ fixed, we define $\widetilde{T}_R g(x) = \sup_{0<\varepsilon\le R}|\widetilde{T}_\varepsilon g(x)|$. Since as $R\to\infty$ $\widetilde{T}_R g(x)$ increases pointwise to $\widetilde{T}^* g(x)$, by monotonic convergence theorem, to prove (8) we only need to prove that

$$\|\widetilde{T}_R g\|_{L^p(\mathbb{T}^n)} \le C\|g\|_{H^p(\mathbb{T}^n)} \tag{9}$$

with a constant C independent of R and $g(x)$.

By Lemma 4, $g(x) = \sum c_k a_k(x)$, where each a_k is a $[n(1/p-1)]+2n$ atom and $\sum |c_k|^p \cong \|g\|^p_{H^p(\mathbb{T}^n)}$.

We let

$$\Psi(x) = \prod_{j=1}^n (1-4x_j^2)_+,$$

where

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x)\ge 0, \\ 0 & \text{if } f(x) < 0.\end{cases}$$

For positive integers M and N, we denote the cube $[-N/2M, N/2M)^n$ by NQ/M. Noting that $\widetilde{T}_R g(x)$ is a periodic function, for large N we have

$$\|\widetilde{T}_R g\|^p_{L^p(\mathbb{T}^n)} \cong N^{-n}\int_{NQ/2} |\widetilde{T}_R g(x)|^p\,dx.$$

Since on $NQ/2$, there exists a constant $C>0$ such that $\Psi(x/N)\ge C$, it is easy to see that

$$\|\widetilde{T}_R g\|^p_{L^p(\mathbb{T}^n)} \cong N^{-n}\int_{NQ/2} |\Psi(x/N)\widetilde{T}_R g(x)|^p\,dx.$$

By Lemma 6 and the assumption of the theorem, we have

$$\begin{aligned}\|\widetilde{T}_R g\|^p_{L^p(\mathbb{T}^n)} &\le CN^{-n}\int_{\mathbb{R}^n} |T^*(g\Psi^{1/N})(x)|^p\,dx \\ &\quad + cN^{-n}\int_{NQ/2}\sup_{0<\varepsilon\le R}|J_{N,\varepsilon}(x)|^p\,dx \\ &\le CN^{-n}\|g\Psi^{1/N}\|^p_{H^p(\mathbb{R}^n)} + o(1), \quad \text{as } N\to\infty.\end{aligned}$$

Now it suffices to show that for odd N

$$\liminf_{N\to\infty} N^{-n}\|g\Psi^{1/N}\|^p_{H^p(\mathbb{R}^n)} \le C\|g\|^p_{H^p(\mathbb{T}^n)}. \tag{10}$$

By Lemma 4, we only need to prove that for any $(p, 2, s)$ periodic atom $a(x)$ with support in $B(x_0, \rho) \subset Q$,

$$N^{-n}\|a\Psi^{1/N}\|^p_{H^p(\mathbb{R}^n)} \le C, \tag{11}$$

where C is a constant independent of $a(x)$ and N.

By the definition, we have

$$\begin{aligned} N^{-n}\|a\Psi^{1/N}\|^p_{H^p(\mathbb{R}^n)} &\cong N^{-n}\int_{\mathbb{R}^n}\sup_{0<t<\infty}\left|\int_{\mathbb{R}^n}\Psi(x/N)a(x)\Phi_t(y-x)\,dx\right|^p dy \\ &= N^{-n}\int_{\mathbb{R}^n}\sup_{0<t<\infty}\left|\int_{\mathbb{R}^n}\prod_{j=1}^n(1-4x_j^2/N^2)_+a(x)\Phi_t(y-x)\,dx\right|^p dy \\ &= N^{-n}\int_{\mathbb{R}^n}\sup_{0<t<\infty}\left|\int_{|x_j|<N/2}\left\{\prod_{j=1}^n(1-4x_j^2/N^2)a(x)\right\}\Phi_t(y-x)\,dx\right|^p dy. \end{aligned}$$

Now we write $N = 2m+1$. Then, up to a set of measure 0, the set $\{x \in \mathbb{R}^n : |x_j| < m + 1/2,\ j = 1, 2, \dots, n\}$ is the union of the disjoint sets $\{Q + k : k = (k_1, \dots, k_n),\ -m \le k_j \le m,\ j = 1, 2, \dots, n\} = \{Q_k\}$, where the k_j's are integers. Now the last integral above is bounded by

$$I_m = Cm^{-n}\sum_{-m\le k_j\le m}\int_{\mathbb{R}^n}\sup_{0<t<\infty}\left|\int_{Q_k}\left\{\prod_{j=1}^n(1-4x_j^2/N^2)a(x)\right\}\Phi_t(y-x)\,dx\right|^p dy.$$

Noting that $a(x)$ is a periodic function, we easily see that $\chi_{Q_k}(x)a(x)$ is an atom with support in Q_k, where χ_{Q_k} is the characteristic function of Q_k. Also since on Q_k, $\prod_{j=1}^n(1-4x_j^2/N^2)$ is a polynomial of degree $2n$ which is bounded by 1, clearly

$$\alpha(x) = \prod_{j=1}^n(1-4x_j^2/N^2)\chi_{Q_k}(x)a(x)$$

is a $(p, 2, [n(1/p-1)])$ atom on $\mathbb{R}^n$. So the above integral I_m is bounded by

$$Cm^{-n}\sum_{-m\le k_j\le m}\|\alpha\|_{H^p(\mathbb{R}^n)} \le C.$$

Theorem 2 is proved.

Following the proof on page 133 in [LL], we now easily obtain an improvement of Theorem 2 in [LL].

Theorem 3. *Let $0 < p \le 1$, and let $1 \le d < n$ be an integer. Suppose that λ is a bounded and continuous function on $\mathbb{R}^n$. If λ is a maximal multiplier on $H^p(\mathbb{R}^n)$ $(H^p(\mathbb{T}^n))$, then the restriction of λ to $\mathbb{R}^d$ is a maximal multiplier on $H^p(\mathbb{R}^d)$ $(H^p(\mathbb{T}^d))$.*

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