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ON THE EXISTENCE OF PERIODIC SOLUTIONS FOR NONCONVEX-VALUED DIFFERENTIAL INCLUSIONS IN $\mathbb{R}^N$

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ABSTRACT. In this paper we investigate the existence of periodic solutions for differential inclusions with nonconvex-valued orientor field. Using a tangential condition and directionally continuous selectors, we establish the existence of periodic trajectories.

1. INTRODUCTION

In this paper, we investigate the following multivalued periodic problem defined on $[0, b] \times \mathbb{R}^N$:

$$\left\{ \begin{array}{c} \dot{x}(t) \in F(t, x(t)) \text{ a.e.} \\ x(0) = x(b). \end{array} \right\} \tag{1}$$

Here $F : T \times \mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\varnothing\}$ is a set-valued vector field (orientor field), which has closed but not necessarily convex values. All earlier results on (1) assumed that the orientor field is convex-valued. We refer to the works of Haddad-Lasry [6] (Theorem B-II-1), Aubin-Cellina [1] (Theorem 4, p. 237), Macki-Nistri-Zecca [10] and Plaskacz [14] (Theorem 4.5).

In this paper using directionally continuous selectors for the orientor field $F(t, x)$ (cf. Bressan [3]) and a Nagumo type tangential condition, analogous to the one employed in [1], [6] and [14], we are able to establish the existence of solutions for (1). The approach of Macki-Nistri-Zecca [10] was based on degree-theoretic techniques.

2. PRELIMINARIES

In what follows by $P_{k(c)}(\mathbb{R}^N)$ we will denote the collection of all nonempty, compact (and convex) subsets of $\mathbb{R}^N$. A multifunction (set-valued function)

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$F : T \times \mathbb{R}^N \to P_k(\mathbb{R}^N)$ is said to be measurable, if for all $z \in \mathbb{R}^N$, $(t, x) \to d(z, F(t, x)) = \inf\{\|z - v\| : v \in F(t, x)\}$ is measurable.

Let V, W be Hausdorff topological spaces and $G : Y \to 2^Z\backslash\{\varnothing\}$. We say that $G(\cdot)$ is lower semicontinuous (l.s.c.) (resp. upper semicontinuous (u.s.c.)), if for all $U \subseteq W$ open the set $G^-(U) = \{v \in V : G(v) \cap U \neq \varnothing\}$ (resp. $G^+(U) = \{v \in V : G(v) \subseteq U\}$) is open in V. We will say that $G(\cdot)$ has an open graph, if the set $\operatorname{Gr} G = \{(v, w) \in V \times W : w \in G(v)\}$ is open in $V \times W$ furnished with the product topology.

Let K be a nonempty and closed subset of $\mathbb{R}^N$ and let $x \in K$. The Bouligand tangent cone to K at x, $T_K(x)$, is defined by

$$T_K(x) = \left\{ v \in \mathbb{R}^N : \liminf_{\lambda \downarrow 0} \frac{d(x + \lambda v, K)}{\lambda} = 0 \right\}.$$

It is immediate from this definition that $T_K(x)$ is a closed, but not necessarily convex, cone in $\mathbb{R}^N$. It is convex if K is convex or more generally locally convex at x.

Now let $T = [0, b]$ and let $K : T \to P_k(\mathbb{R}^N)$. Let $(t, x) \in \operatorname{Gr} K = \{(s, y) \in T \times \mathbb{R}^N : y \in K(s)\}$ (the graph of K). The "contingent derivative" of $K(\cdot)$ at (t, x), $DK(t, x)$, is the set-valued map whose graph is the Bouligand tangent cone to the graph of $K(\cdot)$ at (t, x); i.e. $v \in DK(t, x)(\tau)$ if and only if $(\tau, v) \in T_{\operatorname{Gr} K}(t, x)$ (see Aubin-Cellina [1], Chapter 4).

Finally recall that on $P_k(\mathbb{R}^N)$ we can define a metric, known in the literature as the Hausdorff metric, by setting

$$h(A, B) = \max \left[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right].$$

Recall that $(P_k(\mathbb{R}^N), h)$ is a Polish space (i.e. a complete, separable metric space) and $P_{kc}(\mathbb{R}^N)$ is a closed subset of it. We will say that a multifunction $K : T \to P_k(\mathbb{R}^N)$ is Hausdorff-Lipschitz (h-Lipschitz), if $h(K(t'), K(t)) \leq k|t' - t|$ for some $k > 0$ and all $t, t' \in T$.

3. Auxiliary results

In this section we prove some auxiliary results that we will need in the proof of our main theorem (Section 4).

We start with a Scorza-Dragoni type property for measurable orientor fields $F(t, x)$, which are l.s.c. in x.

Proposition 3.1. *If $T = [0, b]$, X is a metric space, Z is a Polish space and $F : T \times X \to 2^Z\backslash\{\varnothing\}$ is a multifunction with closed values such that*

(1) *$(t, x) \to F(t, x)$ is measurable,*
(2) *$x \to F(t, x)$ is l.s.c.,*

then for every $\varepsilon > 0$, there exists $T_\varepsilon \subseteq T$ compact with $\lambda(T_\varepsilon^c) < \varepsilon$ such that $F|_{T_\varepsilon \times X}$ is l.s.c. (here $\lambda(\cdot)$ denotes the Lebesgue measure on T).

Proof. Consider the function $\gamma : T \times X \times Z \to \mathbb{R}_+$ defined by $\gamma(t, x, z) = d(z, F(t, x))$. Because of the measurability hypothesis on $F(t, x)$, $(t, x) \to \gamma(t, x, z)$ is measurable, while clearly $z \to \gamma(t, x, z)$ is continuous. So $(t, x, z) \to \gamma(t, x, z)$ is measurable. Furthermore, if $(x_n, z_n) \to (x, z)$ in

$X \times Z$, then we have

$$\begin{aligned} d(z_n, F(t, x_n)) &= d(z_n, F(t, x_n)) - d(z, F(t, x_n)) + d(z, F(t, x_n)) \\ &\leq \|z_n - z\| + d(z, F(t, x_n)) \\ \Rightarrow \overline{\lim}\, d(z_n, F(t, x_n)) &\leq \overline{\lim}\, d(z, F(t, x_n)) \leq d(z, F(t, x)), \end{aligned}$$

the last inequality being a consequence of the hypothesis that $x \to F(t, x)$ is l.s.c. (see for example Aubin-Cellina [1], Corollary 1, p. 52). So $(t, x, z) \to d(z, F(t, x))$ is a measurable integrand which is u.s.c. in (x, z). Then define

$$\gamma_k(t, x, z) = \sup[\gamma(t, x, z) - k d_X(x, x') - k d_Z(z, z') : x' \in X,\ z' \in Z]$$

(here $d_X(\cdot, \cdot)$ denotes the metric on X and $d_Z(\cdot, \cdot)$ the metric on Z).

From Lemma 2.1 of Hiai-Umegaki [7], we know that $t \to \gamma_k(t, x, z)$ is measurable, while it is easy to see that $(x, z) \to \gamma_k(t, x, z)$ is k-Lipschitz and $\gamma_k \downarrow \gamma$ as $k \to \infty$ (see for example Bertsekas-Shreve [2], Lemma 7.7, p. 125). Given $\varepsilon > 0$, apply the classical Scorza-Dragoni theorem to find $T_\varepsilon \subseteq T$ compact with $\lambda(T_\varepsilon^c) < \varepsilon$ such that $\gamma_k|_{T_\varepsilon \times X \times Z}$ is continuous for all $k \geq 1$. Since $\gamma_k \downarrow \gamma$, we get that $\gamma|_{T_\varepsilon \times X \times Z}$ is u.s.c. (cf. Bertsekas-Shreve [2], Lemma 7.14, p. 147). So $d(z, F(\cdot, \cdot))|_{T_\varepsilon \times X}$ is u.s.c. $\Rightarrow F|_{T_\varepsilon \times X}$ is l.s.c. (see Klein-Thompson [8]). $\square$

In the next result $\overset{\circ}{B}_\varepsilon(x) = \{y \in X : d_X(y, x) < \varepsilon\}$ (the open ball of radius ε, centered at $x \in X$).

Proposition 3.2. *If X is a metric space, $h : X \to \mathbb{R}^N$ satisfies $\|h(x)\| \leq M$ for all $x \in X$ and we define $G(x) = \bigcap_{\varepsilon > 0} \overline{\text{conv}}\, h(\overset{\circ}{B}_\varepsilon(x))$, then $x \to G(x)$ is u.s.c. from X into $P_{kc}(\mathbb{R}^N)$.*

Proof. Since $G(\cdot)$ is $P_{kc}(\mathbb{R}^N)$-valued, to establish its upper semicontinuity, it is enough to show that for every $x^* \in \mathbb{R}^N$,

$$x \to \sigma(x^*, G(x)) = \sup\{\langle x^*, v\rangle_{\mathbb{R}^N} : v \in G(x)\}$$

is u.s.c. (cf. Klein-Thompson [8]; here $\langle \cdot, \cdot \rangle_{\mathbb{R}^N}$ denotes the Euclidean inner product in $\mathbb{R}^N$). So fix $x^* \in \mathbb{R}^N$. We will show that for every $\lambda \in \mathbb{R}$, the level set $U_\lambda = \{x \in X : \lambda \leq \sigma(x^*, G(x))\}$ is closed. To this end let $\{x_n\}_{n \geq 1} \subseteq U_\lambda$ and assume that $x_n \to x$ in X. Find $y_n \in G(x_n)$ such that $\lambda \leq \langle x^*, y_n \rangle_{\mathbb{R}^N}$. Let $\varepsilon_n = 2 d_X(x_n, x) \downarrow 0$ as $n \to \infty$. Then we have $y_n \in G(\overset{\circ}{B}_{\varepsilon_n}(x))$. Since $\|y_n\| \leq M$, $n \geq 1$, by passing to a subsequence if necessary we may assume that $y_n \to y$. Clearly then $y \in G(x)$ and $\langle x^*, y_n \rangle_{\mathbb{R}^N} \to \langle x^*, y_n \rangle_{\mathbb{R}^N} \Rightarrow \lambda \leq \langle x^*, y \rangle_{\mathbb{R}^N}$ and so $\lambda \leq \sigma(x^*, G(x))$. Therefore $\lambda \in U_\lambda$, which implies that $x \to \sigma(x^*, G(x))$ is u.s.c. $\Rightarrow x \to G(x)$ is u.s.c. $\square$

For the next auxiliary result we will need the following hypothesis.

$H(K)$: $K : T \to P_{kc}(\mathbb{R}^N)$ is an h-Lipschitz multifunction.

Proposition 3.3. *If hypothesis $H(K)$ holds, $G : T \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that for all $(t, x) \in \operatorname{Gr} K$, $t < b$, $G(t, x) \subset DK(t, x)(1)$ and $x : T \to \mathbb{R}^N$ is an absolutely continuous function such that $\dot{x}(t) \in G(t, p(t, x(t)))$*

a.e., $x(0) \in K(0)$ *(here* $p(t, x) = \text{proj}(x; K(t))$*, the metric projection on* $K(t)$*), then for all* $t \in T$*,* $x(t) \in K(t)$*.*

Proof. Let $\varphi(t) = d(x(t), K(t))$. Using hypothesis $H(K)$, we can easily check that $\varphi(\cdot)$ is absolutely continuous on T, with $\varphi(0) = 0$. So to prove our proposition, we only need to show that $\dot{\varphi}(t) \le 0$ a.e.

To this end, let $t \in [0, b)$ be a point at which both $\dot{x}(\cdot)$ and $\dot{\varphi}(\cdot)$ exist. Then we have

$$\begin{aligned}\frac{\varphi(t+h) - \varphi(t)}{h} &= \frac{d(x(t+h), K(t+h)) - d(x(t), K(t))}{h} \\ &= \frac{d(x(t) + h\dot{x}(t) + o(h), K(t+h)) - d(x(t), K(t))}{h} \\ &\le \frac{\|o(h)\|}{h} + \frac{d(x(t) + h\dot{x}(t), K(t+h)) - d(x(t), K(t))}{h}.\end{aligned}$$

Observe that

$$\begin{aligned}&\frac{d(x(t) + h\dot{x}(t), K(t+h)) - d(x(t), K(t))}{h} \\ &\quad \le \frac{1}{h}[\|x(t) - p(t, x(t))\| + d(p(t, x(t)) + h\dot{x}(t), K(t+h)) - d(x(t), K(t))] \\ &\quad = \frac{1}{h} d(p(t, x(t)) + h\dot{x}(t), K(t+h)) \\ &\quad = d\left(\dot{x}(t), \frac{K(t+h) - p(t, x(t))}{h}\right).\end{aligned}$$

Since by hypothesis $\dot{x}(t) \in G(t, p(t, x(t))) \subseteq DK(t, p(t, x(t)))(1)$ a.e. on $[0, b)$, from Aubin-Cellina [1], pp. 190–191, we have that

$$\begin{aligned}&\varliminf_{\lambda \downarrow 0} d\left(\dot{x}(t), \frac{K(t+h) - p(t, x(t))}{h}\right) = 0 \text{ a.e.} \\ &\qquad \Rightarrow \dot{\varphi}(t) \le 0 \text{ a.e.} \\ &\qquad \Rightarrow \varphi(t) = 0 \text{ for all } t \in T \text{ and so } x(t) \in K(t) \text{ for all } t \in T. \quad \square\end{aligned}$$

The next auxiliary result establishes a useful property of lower semicontinuous multifunctions and will lead us to a new lower semicontinuity result concerning the intersection of two multifunctions.

Proposition 3.4. *If* Z *is a Hausdorff topological space,* $F : Z \to 2^{\mathbb{R}^N} \setminus \{\varnothing\}$ *is l.s.c. with convex values and* $\mathring{B}(x, \hat{r}) = \{y \in \mathbb{R}^N : \|y - x\| < \hat{r}\} \subseteq F(z_0)$*, then for all* $r \in (0, \hat{r})$*, there exists an open neighborhood* U *of* z_0 *such that* $\mathring{B}(x, r) \subseteq F(z)$ *for all* $z \in U$*.*

Proof. Let $r \in (0, \hat{r})$ and take $r', \varepsilon > 0$ such that $0 < r < r' < \hat{r}$ and $0 < \varepsilon < r' - r$. We have $\mathring{B}(x, r') \subseteq F(z_0)$. Let $\theta(z) = h^*(B(x, r'), F(z)) = \sup[d(v, F(z)) : v \in B(x, r')]$. We know (cf. Aubin-Cellina [1], Theorem 5, p. 52) that $\theta(\cdot)$ is u.s.c. and $\theta(z_0) = 0$. So we can find U an open neighborhood of z_0 such that $\theta(z) < \varepsilon$ for all $z \in U$. Since by hypothesis $F(\cdot)$ is convex-valued from Moreau [11], we know that

$$\begin{aligned} d(x, \mathbb{R}^N \backslash \mathring{B}(x, r')) - d(x, \mathbb{R}^N \backslash F(z)) &\leq h^*(B(x, r'), F(z)) \\ &= \theta(z) < \varepsilon \quad \text{for all } z \in U, \end{aligned}$$

$$\Rightarrow d(x, \mathbb{R}^N \backslash \mathring{B}(x, r')) - \varepsilon \leq d(x, \mathbb{R}^N \backslash F(z)),$$
$$\Rightarrow r < r' - \varepsilon \leq d(x, \mathbb{R}^N \backslash F(z)) \quad \text{for all } z \in U.$$

So we conclude that for all $z \in U$, $\mathring{B}(x, r) \subseteq F(z)$. □

Having this proposition, we can now prove a lower semicontinuity result for the intersection of two multifunctions.

Proposition 3.5. *If Z is a Hausdorff topological space, $H_1, H_2 : Z \to 2^{\mathbb{R}^N} \backslash \{\varnothing\}$ are l.s.c. multifunctions such that for all $z \in Z$, $H_2(z)$ is open and convex and $H_1(z) \cap H_2(z) \neq \varnothing$, then $z \to H(z) = H_1(z) \cap H_2(z)$ is l.s.c.*

Proof. We claim that $\operatorname{Gr} H_2$ is open. Indeed let $(z, v) \in \operatorname{Gr} H_2$. Then $v \in H_2(z)$ and so we can find $\hat{r} > 0$ such that $\mathring{B}(v, \hat{r}) \subseteq H_2(z)$. Let $r < \hat{r}$ and apply Proposition 3.4 to get U an open neighborhood of z such that $\mathring{B}(v, r) \subseteq H_2(z')$ for all $z' \in U$. Set $W = U \times \mathring{B}(v, r)$. Clearly this is an open neighborhood of $(z, v) \in Z \times \mathbb{R}^N$ and for every $(z', v') \in W$ we have $v' \in \mathring{B}(v, r) \subseteq H_2(z')$. So $(z', v') \in \operatorname{Gr} H_2$ for all $(z', v') \in W \Rightarrow \operatorname{Gr} H_2$ is open and so by the lemma in Papageorgiou [13], we conclude that $z \to H_1(z) \cap H_2(z)$ must be l.s.c. □

Let Γ be a cone in $\mathbb{R}^m$ and X a metric space. Following Bressan [3], we say that the single-valued map $f : \mathbb{R}^m \to X$ is Γ-continuous at $v \in \mathbb{R}^m$, if and only if $f(v_n) \to f(v)$ when $v_n \to v$ in $\mathbb{R}^m$ and $v_n - v \in \Gamma$ for all $n \geq 1$ (equivalently for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d_X(f(v), f(v')) < \varepsilon$ when $\|v - v'\| < \delta$, $v' - v \in \Gamma$). We will say that $f(\cdot)$ is Γ-continuous, if it is Γ-continuous at every $v \in \mathbb{R}^m$. Let $M > 0$ and on $\mathbb{R} \times \mathbb{R}^N$ consider the cone $\Gamma^M = \{(t, x) \in \mathbb{R} \times \mathbb{R}^N : \|x\| \leq Mt\}$. A map $h : T \times \mathbb{R}^N \to \mathbb{R}^N$ is said to be Scorza-Dragoni Γ^M-continuous, if for every $\varepsilon > 0$, we can find $T_\varepsilon \subseteq T$ compact such that $\lambda(T_\varepsilon^c) < \varepsilon$ and $h|_{T_\varepsilon \times \mathbb{R}^N}$ is Γ^M-continuous.

Now let $h : T \times \mathbb{R}^N \to \mathbb{R}^N$ be a Scorza-Dragoni Γ^M-continuous map such that $\|h(t, x)\| \leq M$ for all $(t, x) \in T \times \mathbb{R}^N$. Let $G(t, x) = \bigcap_{\varepsilon > 0} \overline{\operatorname{conv}} h(\mathring{B}_\varepsilon(t, x))$ (cf. Proposition 3.2). We consider the following two Cauchy problems:

$$\left\{ \begin{array}{c} \dot{x}(t) \in G(t, x(t)) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\} \tag{2}$$

and

$$\left\{ \begin{array}{c} \dot{x}(t) = h(t, x(t)) \text{ a.e.} \\ x(0) = x_0. \end{array} \right\} \tag{3}$$

We have the following result relating these two problems.

Proposition 3.6. *If $x : T \to \mathbb{R}^N$ is a solution of* (2), *then $x(\cdot)$ also solves* (3).

Proof. From Lusin's theorem and the fact that $h(\cdot, \cdot)$ is Scorza-Dragoni Γ^M-continuous, we can find $T_n \subseteq T$, $n \geq 1$, measurable sets such that $\dot{x}|_{T_n}$ is continuous, $h|_{T_n \times \mathbb{R}^N}$ is Γ^M-continuous, $\dot{x}(t) \in G(t, x(t))$ for all $t \in T_n$ and $\lambda(T \setminus \bigcup_{n \geq 1} T_n) = 0$. Also invoking Lebesgue's density theorem (see Oxtoby [12]), we can find sets $N_n \subseteq T_n$, with $\lambda(N_n) = 0$, such that every $t \in T_n \setminus N_n$ is a density point for T_n. Next let $t \in T_n \setminus N_n$. We can find $t_m \in T_n \setminus N_n$, $t_m > t$, $m \geq 1$, and $t_m \downarrow t$ as $m \to \infty$. So $\dot{x}(t_m) \to \dot{x}(t)$. Also because by hypothesis $\|h(t, x)\| \leq M$, we have $|G(t, x)| = \sup\{\|v\| : v \in G(t, x)\} \leq M$ and so $\|x(t) - x(s)\| \leq M|t - s|$ for all $t, s \in T$. Let $\varepsilon > 0$. We have $\dot{x}(t_m) \in G(t_m, x(t_m)) \subseteq h(t_m, x(t_m)) + (\varepsilon/2)\mathring{B}_1$, where $\mathring{B}_1 = \{y \in \mathbb{R}^N : \|y\| < 1\}$. Since $h|_{T_n \times \mathbb{R}^N}$ is Γ^M-continuous, we can find $m_0(\varepsilon) \geq 1$ such that for $m \geq m_0$ we have

$$\begin{aligned} &\|h(t_m, x(t_m)) - h(t, x(t))\| < \frac{\varepsilon}{2} \\ &\qquad \Rightarrow h(t_m, x(t_m)) \in h(t, x(t)) + \frac{\varepsilon}{2}\mathring{B}_1. \end{aligned}$$

Hence for $m \geq m_0(\varepsilon)$ we have

$$\begin{aligned} &\dot{x}(t_m) \in h(t, x(t)) + \varepsilon \mathring{B}_1 \\ &\qquad \Rightarrow \dot{x}(t) \in h(t, x(t)) + \varepsilon B_1. \end{aligned}$$

Let $\varepsilon \downarrow 0$, to get that for all $t \in \widehat{T} = \bigcup_{n \geq 1}(T_n \setminus N_n)$, $\lambda(T \setminus \widehat{T}) = 0$, we have $\dot{x}(t) = h(t, x(t))$, $x(0) = x_0 \Rightarrow x(\cdot)$ is a solution of (3). $\square$

4. Main theorem

In this section we state and prove our main result, concerning the existence of solutions for problem (1). For this we will need the following hypotheses:

$H(F)$: $F : T \times \mathbb{R}^N \to P_k(\mathbb{R}^N)$ is a multifunction such that

(1) $(t, x) \to F(t, x)$ is measurable,
(2) $x \to F(t, x)$ is l.s.c.,
(3) $|F(t, x)| = \sup\{\|v\| : v \in F(t, x)\} \leq M$ for all $(t, x) \in T \times \mathbb{R}^N$.

$H(K)'$: $K : T \to P_{kc}(\mathbb{R}^N)$ is an h-Lipschitz multifunction such that $K(0) \supseteq K(b)$, $(t, x) \to DK(t, x)(1)$ is l.s.c. on $\operatorname{Gr} K \cap ([0, b) \times \mathbb{R}^N)$ and for every $(t, x) \in \operatorname{Gr} K \cap ([0, b) \times \mathbb{R}^N)$ we have $\operatorname{int} DK(t, x)(1) \neq \varnothing$.

Remark. Note that if $K(t) = K \in P_{kc}(\mathbb{R}^N)$, $t \in T$, and $\operatorname{int} K \neq \varnothing$, then hypothesis $H(K)'$ is automatically satisfied. To see this, remark that $DK(t, x)(1) = \operatorname{proj}_{\mathbb{R}^N}([\{1\} \times \mathbb{R}^N] \cap T_{\operatorname{Gr} K}(t, x)) = \operatorname{proj}_{\mathbb{R}^N}([\{1\} \times \mathbb{R}^N] \cap [T_T(x) \times T_K(x)]) = T_K(x)$ and apply Theorem 1 and Proposition 4, pp. 220–221, of Aubin-Cellina [1]. Also note that our lower semicontinuity hypothesis on $(t, x) \to DK(t, x)(1)$ implies that this multifunction is convex-valued (cf. Aubin-Cellina [1]).

H_τ: For all $(t, x) \in \operatorname{Gr} K$, $t < b$, we have $F(t, x) \cap \operatorname{int} DK(t, x)(1) \neq \varnothing$ (tangential condition).

Theorem 4.1. *If hypotheses $H(F)$, $H(K)'$ and H_τ hold, then problem* (1) *admits a solution.*

Proof. From Proposition 3.5, we have that $x \to F(t, x) \cap \operatorname{int} DK(t, x)(1)$ is l.s.c. $\Rightarrow x \to \overline{F(t, x) \cap \operatorname{int} DK(t, x)(1)}$ is l.s.c. Also because of hypothesis $H(F)(1)$, we have that $(t, x) \to \overline{F(t, x) \cap \operatorname{int} DK(t, x)(1)}$ is measurable. Let $H(t, x) = \overline{F(t, x) \cap \operatorname{int} DK(t, x)(1)}$. From Proposition 3.1 we know that given $\varepsilon > 0$, we can find $T_\varepsilon \subseteq T$ compact with $\lambda(T_\varepsilon^c) < \varepsilon$ such that $H|_{T_\varepsilon \times \mathbb{R}^N}$ is l.s.c. So we can apply Theorem 1 of Bressan [3] and get a function $h : T \times \mathbb{R}^N \to \mathbb{R}^N$ such that $h(\cdot, \cdot)$ is Scorza-Dragoni Γ^M-continuous and for all $(t, x) \in T \times \mathbb{R}^N$, $h(t, x) \in H(t, x)$ (recall Γ^M is the cone $\{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N : \|x\| \leq Mt\}$). Then consider the multifunction $G(t, x) = \bigcap_{\varepsilon > 0} \overline{\operatorname{conv}} h(\overset{\circ}{B}_\varepsilon(t, x))$, where $\overset{\circ}{B}_\varepsilon(t, x) = \{(t', x') \in T \times \mathbb{R}^N : |t - t'| < \varepsilon,\ \|x - x'\| < \varepsilon\}$. From Proposition 3.2 we know that $(t, x) \to G(t, x)$ is u.s.c., and clearly from the above definition we have that $G(t, x) \subseteq \overline{\operatorname{conv}} F(t, x) \subseteq DK(t, x)(1)$ for all $(t, x) \in \operatorname{Gr} K$, $t < b$. Then consider the following Cauchy problem:

$$\left\{ \begin{aligned} \dot{x}(t) &\in G(t, p(t, x(t))) \text{ a.e.} \\ x(0) &= z \in K(0). \end{aligned} \right\} \tag{4}$$

It is well known (see for example, DeBlasi-Myjak [4] and Gorniewicz [5]) that the solution set $S(z)$ of (4) is an R_δ-set in $C(T, \mathbb{R}^N)$. In particular, then it is compact and acyclic (with respect to the Čech cohomology; cf. Proposition 3.1 of Gorniewicz [5]). Furthermore from Proposition 3.3, we know that $S(z)(b) = \{x(b) : x \in S(z)\} \subseteq K(b) \subseteq K(0)$. Let $e_b : C(T, \mathbb{R}^N) \to \mathbb{R}^N$ be the evaluation at b map; i.e. $e_b(x) = x(b)$. Let $R = e_b \circ S : K(0) \to P_k(K(0))$. This is a pseudo-acyclic map in the sense of Lasry-Robert [9] (see Definition 5), and so by Theorem 7 of that paper we can find $z \in R(z)$. Let $x \in S(z)$ such that $x(0) = x(b) = z$. Then $\dot{x}(t) \in G(t, p(t, x(t)))$ a.e., $x(0) = x(b)$. Recalling that $G(t, x) \subseteq DK(t, x)(1)$ for $(t, x) \in \operatorname{Gr} K$, $t < b$, from Proposition 3.3 we get that $x(t) \in K(t)$ for all $t \in T$. So $\dot{x}(t) \in G(t, x(t))$ a.e., $x(0) = x(b)$. Invoking Proposition 3.6, we get $\dot{x}(t) = h(t, x(t)) \in F(t, x(t))$ a.e., $x(0) = x(b)$; i.e. $x(\cdot)$ solves (1). □

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