COUNTABLY ADDITIVE HOMOMORPHISMS
BETWEEN VON NEUMANN ALGEBRAS

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ABSTRACT. Let M and N be von Neumann algebras where M has no abelian direct summand. A $*$-homomorphism $\pi: M \to N$ is said to be countably additive if $\pi(\sum_1^\infty e_n) = \sum_1^\infty \pi(e_n)$, for every sequence (e_n) of orthogonal projections in M. We prove that a $*$-homomorphism $\pi: M \to N$ is countably additive if and only if $\pi(e \vee f) = \pi(e) \vee \pi(f)$ for every pair of projections e and f of M. A corollary is that if, in addition, M has no Type I_2 direct summands, then every lattice morphism from the projections of M into the projections of N is a σ-lattice morphism.

Let M and N be von Neumann algebras. A $*$-homomorphism $\pi: M \to N$ is said to be *countably additive* if $\pi(\sum e_n) = \sum \pi(e_n)$ for every *sequence* (e_n) of orthogonal projections in M. Thus countable additivity is equivalent to weak-$*$ continuity on all σ-finite von Neumann subalgebras of M.

It is well known [3, 8] (see also [9, V. 5.1]) that a $*$-homomorphism $\pi: M \to N$ is automatically weak-$*$ continuous when

(a) M is Type II_1 and σ-finite and N acts on a separable Hilbert space, or

(b) M is properly infinite and M and N are σ-finite.

It is worth noting that, in either case, it is not enough to know only that M is σ-finite. For instance (see the argument of [9, V. 5.2], for example) when M is Type II, with an infinite-dimensional centre, then M has weak-$*$ discontinuous factor representations; and when $M = B(H)$, where H is a separable (or otherwise) Hilbert space, then each nontrivial representation of the Calkin algebra $B(H)/K(H)$ induces a weak-$*$ discontinuous representation of M.

The purpose of this paper is to show that, free of all cardinality qualifications and all (except trivial) restrictions upon the type of the von Neumann algebras in question, a natural algebraic condition characterises countably additive $*$-homomorphisms. We prove:

Theorem. *Let* $\pi: M \to N$ *be a* $*$-*homomorphism between von Neumann algebras, where* M *has no abelian direct summand. Then the following are equivalent.*

(i) $\pi(e \vee f) = \pi(e) \vee \pi(f)$, *for all projections e and f.*
(ii) π *is countably additive.*

We remark that the Theorem fails when M is abelian and infinite dimensional because then, as is easy to see, condition (ii) need not be true whereas (i) is automatically satisfied.

The naturality of the condition (i) of the Theorem may be seen through the axiomatic foundations of quantum mechanics. Given projections e and f of M we have $[e + 1 - (e \vee f)] \vee [f + 1 - (e \vee f)] = 1$. Thus in the case when $\pi(1) = 1$, the condition (i) is equivalent to (for projections e and f of M)

$$e \vee f = 1 \quad \text{implies} \quad \pi(e) \vee \pi(f) = 1.$$

This has the following completely natural physical interpretation:

the truth of the 'proposition' "e or f" implies the truth of "$\pi(e)$ or $\pi(f)$".

Thus, apart from any independent mathematical interest, the Theorem might find some use in the foundations of quantum mechanics.

We need two preparatory lemmas.

Lemma 1. *Let ρ_1 be a singular state of l^∞, and let ρ_2 be a pure state of $M_n(\mathbb{C})$, where $2 \leq n < \infty$. Then there exist projections e and f in $l^\infty \otimes M_n(\mathbb{C})$ for which $e \vee f = 1$ and $(\rho_1 \otimes \rho_2)(e) = (\rho_1 \otimes \rho_2)(f) = 0$.*

Proof. This was proved in [5, Example 3.2]. See also [1, Lemma 1].

Lemma 2 (Fillmore [4], Kadison [6]). *Let M be a von Neumann algebra, and let (f_n) be a sequence of orthogonal projections in M.*

(i) *If M is Type I_n, where $2 \leq n < \infty$, then there exist orthogonal and equivalent projections $e_1, \ldots, e_n$ in M with sum 1 such that $e_i f_j = f_j e_i$, for all i and j.*

(ii) *If M has no finite Type I direct summand, then there exists in M a projection $e \sim 1 - e$ such that $e f_j = f_j e$, for all j.*

Proof. The cases of (i) and, when M is Type II_1, (ii) are proved in [6, Lemma 3.6 and Corollary 3.15]. As the von Neumann subalgebra of M, $W^*(\{f_n : n \in \mathbb{N}\}) \simeq l^\infty$, generated by the f_n is singly generated, the case of (ii) when M is properly infinite follows immediately from [4, Theorem 3].

Proof of Theorem. (ii) $\Rightarrow$ (i). Let the $*$-homomorphism $\pi: M \to N$ be countably additive, and let e and f be projections in M. Then $\pi(e) \vee \pi(f) \leq \pi(e \vee f)$. Let ρ be a normal state of N such that $\rho(\pi(e) \vee \pi(f)) = 0$. By spectral theory, there is an increasing sequence (p_n) of projections in M converging strongly to $e \vee f$ where, for each n, $e + f \geq \alpha_n p_n$ for some real number $\alpha_n > 0$. But then

$$\rho \pi(e \vee f) = \rho \lim \pi(p_n) = \lim \rho \pi(p_n) = 0.$$

Therefore $\pi(e) \vee \pi(f) = \pi(e \vee f)$.

(i) $\Rightarrow$ (ii). Assume (i) and, without loss of generality, that $\pi(1) = 1$. Suppose first that M contains orthogonal and equivalent projections $e_1, \ldots, e_n$ such that $e_1 + \cdots + e_n = 1$, for some n where $2 \leq n < \infty$. Put $e = e_1$. Via a suitable choice of $n \times n$ matrix units (u_{ij}), there is (see [9, IV 1.8]) a surjective $*$-isomorphism

$$\varphi: M \to (eMe) \otimes M_n(\mathbb{C}).$$

In turn $(\pi(u_{ij}))$ is a system of $n \times n$ matrix units for N, inducing surjective $*$-isomorphism

$$\psi: N \to (fNf) \otimes M_n(\mathbb{C}),$$

where $f = \pi(e)$, and resulting in the commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\ \pi\ } & N \\
\varphi \downarrow & & \downarrow \psi \\
(eMe) \otimes M_n(\mathbb{C}) & \xrightarrow{\pi_1 \otimes \pi_2} & (fNf) \otimes M_n(\mathbb{C})
\end{array}
$$

where $\pi_1: eMe \to fNf$ satisfies $\pi_1(a) \otimes 1_n = \psi\pi(a)$ and π_2 is an automorphism of $M_n(\mathbb{C})$. In particular (as ψ is normal, being an isomorphism), we see that $\pi|eMe$ is countably additive if π_1 is countably additive. We prove the latter.

In order to derive a contradiction, suppose that π_1 is not countably additive. Then there is a sequence (p_n) of orthogonal projections in eMe such that

$$\sum_1^\infty p_n = e \quad \text{and} \quad \sum_1^\infty \pi_1(p_n) \neq \pi_1(e) = f.$$

Choose a normal state ρ_1 of fNf such that $\rho_1(\sum_1^\infty \pi_1(p_n)) = 0$ and let ρ_2 be a pure state of $M_n(\mathbb{C})$. Then $\rho_1 \otimes \rho_2$ is a normal state of $(fNf) \otimes M_n(\mathbb{C})$. In addition, $\rho_1\pi_1$ is singular on the von Neumann subalgebra of M, $W^*(\{p_n: n \in \mathbb{N}\}) \simeq l^\infty$, generated by the p_n and $\rho_2\pi_2$ is a pure state of $M_n(\mathbb{C})$. Hence, by Lemma 1 applied to the state

$$(\rho_1 \otimes \rho_2)(\pi_1 \otimes \pi_2) = (\rho_1\pi_1) \otimes (\rho_2\pi_2),$$

there exist projections p and q in $(eMe) \otimes M_n(\mathbb{C})$ such that

$$p \vee q = 1 \quad \text{and} \quad (\rho_1 \otimes \rho_2)(\pi_1 \otimes \pi_2)(p) = (\rho_1 \otimes \rho_2)(\pi_1 \otimes \pi_2)(q) = 0.$$

Putting now $g = \varphi^{-1}(p)$ and $h = \varphi^{-1}(q)$ we have that $g \vee h = 1$ so that $\pi(g) \vee \pi(h) = 1$, by assumption. On the other hand $\tau = (\rho_1 \otimes \rho_2)\psi$ is a normal state of N with $\tau\pi(g) = \tau\pi(h) = 0$, so that $\tau(\pi(g) \vee \pi(h)) = 0$ in contradiction.

This proves that π is countably additive on e_iMe_i, for each $i = 1, \ldots, n$.

We will proceed to establish (ii) on a case-by-case basis. Let (f_n) be a sequence of orthogonal projections in M.

(a) If M is Type I_n, where $2 \leq n < \infty$ then, by part (i) of Lemma 2, there exist orthogonal and equivalent projections e_i in M with $e_1 + \cdots + e_n = 1$ such that $e_i f_j = f_j e_i$ for all i and j. Since, for each $i = 1, \ldots, n$, π is countably additive (by the above) on e_iMe_i and (e_if_n) is an orthogonal sequence of projections in e_iMe_i, we have

$$\pi(e_i)\pi\left(\sum_1^\infty f_n\right) = \pi\left(e_i \sum_1^\infty f_n\right) = \pi(e_i)\sum_1^\infty \pi(f_n),$$

from which it follows that π is countably additive on M.

(b) If M is finite Type I, then M is an l^∞ direct sum of countably many algebras of the form Type I_n, where $2 \leq n < \infty$. So, in view of (a), it is clearly enough in this case to show that π is countably additive on the centre,

$Z(M)$, of M. We can always choose projections e_1, e_2, e_3 and f in M so that $e_1 + e_2 + e_3 = 1$ with

$$e_1 \sim e_2 \quad \text{and} \quad e_3 \sim f \leq e_1 + e_2.$$

Now the first part of the proof applied to each of these equivalences, in turn, implies that π is countably additive on $e_i M e_i$ and hence on $Z(M)e_i$, for $e = 1, 2, 3$. It follows that π is countably additive on $Z(M)$, as required.

(c) Finally, when M is without finite Type I direct summands, part (ii) of Lemma 2 ensures the existence in M of a projection $e \sim 1 - e$ commuting with all of the f_n and a calculation similar to that performed in (a) establishes countable additivity.

This completes the proof.

If M is von Neumann algebra, then let $P(M)$ denote the projection lattice of M. Let M and N be von Neumann algebras. A function $\varphi \colon P(M) \to P(N)$ is said to be a *lattice morphism* if

(i) $\varphi(e + f) = \varphi(e) + \varphi(f)$, whenever e and f are orthogonal;

(ii) $\varphi(e \vee f) = \varphi(e) \vee \varphi(f)$, for all e and f.

A lattice morphism $\varphi \colon P(M) \to P(N)$ is called a *σ-morphism* if

$$\varphi\left(\bigvee_1^\infty e_n\right) = \bigvee_1^\infty \varphi(e_n) \quad \text{for every sequence } (e_n).$$

One consequence (Corollary 2) of the Theorem is that a lattice morphism $\varphi \colon P(M) \to P(N)$ is nearly always a σ-morphism.

A *$\ast$-Jordan homomorphism* $\pi \colon M \to N$ is a $\ast$-linear map where $\pi(x^2) = \pi(x)^2$, for all x in M. By a result of Størmer [7, Theorem 3.3] each $\ast$-Jordan homomorphism between von Neumann algebras is the sum of a $\ast$-homomorphism and a $\ast$-antihomomorphism. So, the Theorem, being valid for $\ast$-homomorphism and hence (trivially) for $\ast$-antihomomorphisms, is therefore valid for $\ast$-Jordan homomorphisms too. In other words, we have:

Corollary 1. *If $\pi \colon M \to N$ is a $\ast$-Jordan homomorphism between von Neumann algebras M and N where M has no abelian direct summands, then the following are equivalent.*

(i) $\pi(e \vee f) = \pi(e) \vee \pi(f)$ *for all projections e and f.*

(ii) π *is countably additive.*

Corollary 2. *Let M and N be von Neumann algebras where M has neither abelian nor Type I_2 direct summands. Then every lattice morphism $\varphi \colon P(M) \to P(N)$ is a σ-morphism.*

Proof. A lattice morphism $\varphi \colon P(M) \to P(N)$ extends to a $\ast$-Jordan homomorphism $\pi \colon M \to N$ (see [2, Proposition and Corollary 1], for example) which, by Corollary 1, is countably additive. So φ must be a σ-morphism.

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