ON A FIXED POINT THEOREM OF KIRK

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ABSTRACT. Let X be a reflexive Banach space, D an open and bounded subset of X, and $T: \overline{D} \to X$ a continuous mapping which is locally pseudo-contractive on D. Suppose there exists an element $z \in D$ such that $\|z - Tz\| < \|x - Tx\|$ for all x on the boundary of D. Then under the so-called condition (S), T has a fixed point in D. Although this result was proved earlier by Kirk, we show here a much easier approach.

Let X be a (real) Banach space and let D be a subset of X. An operator $T: D \to X$ is said to be *k-pseudo-contractive* $(k > 0)$ (see [8]) if for each $x, y \in D$ and $\lambda > k$

$$(1) \qquad (\lambda - k)\|x - y\| \le \|\lambda(x - y) - (Tx - Ty)\|.$$

For $k = 1$ $(k < 1)$ such mappings are said to be *pseudo-contractive* (respectively, *strongly pseudo-contractive*). One can easily derive the original definition of pseudo-contractive mappings, introduced by Browder [1], which states:

$$(2) \qquad \|x - y\| \le \|(1 + r)(x - y) - r(Tx - Ty)\|$$

for all $x, y \in D$ and all $r > 0$. Furthermore, the semi-inner product formulation (see Kato [5]) of this family of operators appears to be of significant use as well. This means that (2) may be described by

$$\langle Tx - Ty, j \rangle \le \|x - y\|^2$$

for some $j \in J(x - y)$. The mapping $J: X \to 2^{X^*}$ is called the normalized duality mapping which is defined by

$$J(x) = \{ j \in X^*: \langle x, j \rangle = \|x\|^2, \|j\| = \|x\| \}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is easy to verify that nonexpansive mappings are pseudo-contractive. In addition, if the condition (1) holds locally, i.e., if each point $x \in D$ has a neighborhood U such that the restriction of T to U is k-pseudo-contractive with (uniform) constant k, then T is said to be *locally k-pseudo-contractive*.

The purpose of this paper is to provide an alternative proof of Theorem 1 of Kirk [6]. Kirk's proof is certainly ingenious, but it is also quite long

and involved. To the contrary, we offer a different approach which, to our knowledge, appears to be significantly much shorter. Other results are also simplified.

To fix our notation, we will denote the closure and boundary of D by $\overline{D}$ and ∂D respectively, and for $u, v \in X$ we use seg$[u, v]$ to denote the segment $\{tu + (1 - t)v: t \in [0, 1]\}$. We will also use $B(x; r)$ and $\overline{B}(x; r)$ to stand for the open ball $\{z \in X: \|x - z\| < r\}$ and the closed ball $\{z \in X: \|x - z\| \leq r\}$ respectively. We denote the distance between the sets A and B by dist(A, B), and define it by

$$\text{dist}(A, B) = \inf\{\|a - b\|: a \in A, b \in B\}.$$

The results of this paper are formulated either in arbitrary Banach spaces or, for stronger conclusions, in spaces in which the class of locally nonexpansive mappings enjoys some kind of smoothness. To be more precise, suppose G is a bounded open subset of a normed space X and f is an arbitrary locally nonexpansive mapping from G into X. We say that the space X satisfies *condition* (S) if for each $\varepsilon > 0$ sufficiently small, there exists a largest $\xi(\varepsilon) \in (0, \varepsilon]$ such that if $u, v \in G$ for which seg$[u, v] \subset G$, then the conditions $\|u - f(u)\| \leq \xi(\varepsilon)$ and $\|v - f(v)\| \leq \xi(\varepsilon)$ imply $\|m - f(m)\| \leq \varepsilon$ for all $m \in \text{seg}[u, v]$. We should note that ξ also depends on G. Due to a result that implicitly appears in Browder [2] and Göhde [4], it can easily be shown that every uniformly convex space satisfies this condition (S).

Now, we begin with a result for local contraction under an apparently new boundary condition.

Proposition 1. *Let X be a Banach space, let K be a closed subset of X, and let D be an open subset of X with $D \cap K \neq \varnothing$. Suppose $T: \overline{D \cap K} \to K$ is a closed mapping which is a local contraction (with constant k) on $G = D \cap K$, and suppose there exists $z \in G$ such that*

$$(3) \qquad \|z - Tz\| < \|x - Tx\| + (1 - k)\|x - z\| \quad \text{for } x \in \partial_K D.$$

Then $\{T^n(z)\}$ lies in G and converges to a fixed point of T.

Proof. We first prove that seg$[z, Tz] \subset G$. To this end, we assume the contrary. Then there exists $m \in \text{seg}[z, Tz] \cap \partial_K D$ for which seg$[z, m) \subset G$. Since T is a global contraction on seg$[z, m]$, we have

$$\|m - Tm\| \leq \|m - Tz\| + \|Tz - Tm\|$$
$$\leq \|z - Tz\| - \|z - m\| + k\|z - m\|$$
$$= \|z - Tz\| - (1 - k)\|z - m\|.$$

This contradicts (3) since $m \in \partial_K D$. Therefore, seg$[z, Tz] \subset G$. By replacing z by Tz, condition (3) also holds. This means $T^n(z) \in G$ for all $n \in \mathbb{N}$, while a standard argument shows that $\{T^n(z)\}$ is a Cauchy sequence.

Corollary 1. *Let X and K be as in Proposition 1, and let D be a bounded open subset of X. Let $T: \overline{D \cap K} \to K$ be a closed mapping which is locally nonexpansive on $G = D \cap K$ $(\neq \varnothing)$, and suppose there exists $z \in G$ such that*

$$(4) \qquad \|z - Tz\| < \|x - Tx\| \quad \text{for all } x \in \partial_K D.$$

Then $\inf\{\|x - Tx\|: x \in G\} = 0$.

Proof. For each $t \in (0, 1)$, the mapping T_t defined by $T_t(x) = (1-t)z + tT(x)$ is a local contraction on G, and

$$\|z - T_t(z)\| < \|x - T_t(x)\| + (1 - t)\|x - z\|$$

for $x \in \partial_K D$. Then $\{T_t^n(z)\}$ lies in G, and due to the proof of Proposition 1, $\inf\{\|x - T_t(x)\| : x \in G\} = 0$ for each $t \in (0, 1)$. Since D is bounded and also

$$x - T_t(x) = (1 - t)(x - z) + t(x - Tx)$$

for $x \in G$ and $t \in (0, 1)$, it follows that $\inf\{\|x - Tx\| : x \in G\} = 0$.

Before we state the main theorem of this paper, we prove some preliminary facts. For the sake of clarity, we will use G_β and $C(G_\beta)$ to denote the set $\{x \in D \cap K : \|x - Tx\| < \beta\}$ and a connected component of G_β respectively.

Lemma. *Let X be a Banach space which satisfies the condition* (S), *let K be a closed convex subset of X, and let D be a bounded open subset of X with $D \cap K \neq \varnothing$. Suppose $T : D \cap K \to K$ is a local nonexpansive mapping. Furthermore, assume that the set G_{β_0} is nonempty for some positive number β_0 and $\mathrm{dist}(\partial_K G_{\beta_0}, \partial_K D) > 0$. Then there exists a positive number $\beta_1 < \beta_0$ for which*

$$(5) \qquad \mathrm{seg}[u, v] \subset G_{\beta_0} \quad \text{for } u, v \in C(G_{\beta_1}).$$

Proof. Since G_{β_0} is open in K, there exists an open set D_0 in D so that $G_{\beta_0} = D_0 \cap K$. Also, since G_{β_0} is nonempty, there is an element $z \in G_{\beta_0}$ for which condition (4) holds on $\partial_K D_0$. This means that we may select $0 < \alpha_0 < \beta_0$ and $\beta_1 = \xi(\alpha_0)$, where the set $G_{\beta_1} = \{x \in G_{\beta_0} : \|x - Tx\| < \beta_1\}$ is clearly nonempty due to Corollary 1. We shall show now that (5) holds. To see this, let $u \in G_{\beta_1}$ and define the set

$$G_{\beta_1}(u) = \{v \in G_{\beta_1} : \mathrm{seg}[u, v] \subset G_{\beta_0}\}.$$

We may assume, without loss of generality, that G_{β_1} is a connected component of G_{β_0}. Then we claim that $G_{\beta_1}(u) = G_{\beta_1}$. Otherwise, we may assume there exists $w \in G_{\beta_1} \backslash G_{\beta_1}(u)$. Since G_{β_1} is path-connected, there is a path γ in G_{β_1} that joins u with w. This means that there exists $v \in \partial_K G_{\beta_1}(u) \cap G_1$ so that $\mathrm{seg}[u, v] \in \overline{G_{\beta_0}}$. We shall show now that $\mathrm{seg}[u, v] \subset G_{\beta_0}$. To see this, let $x \in \mathrm{seg}[u, v]$. Then, since u and v are in G_{β_1}, the condition (S) implies that $\|x - Tx\| \leq \alpha_0 < \beta_0$, which means $x \in G_{\beta_0}$. Therefore $v \in G_{\beta_1}(u)$, and thus $G_{\beta_1}(u) = G_{\beta_1}$. This completes the proof.

Next, we give a much simpler proof of Theorem 1 of [7], where just a particular case of it will be used in our main theorem.

Proposition 2. *Let X be a reflexive Banach space, and let K and D be as in the previous lemma. Let $T : \overline{D \cap K} \to K$ be a closed mapping which is locally nonexpansive on $G = D \cap K$, and suppose the following conditions hold:*

    (i) *There exists $z \in G$ such that $\|z - Tz\| < \|x - Tx\|$ for all $x \in \partial_K D$.*

    (ii) *X satisfies the condition* (S) *with respect to T.*

Then T has a fixed point in G.

Proof. We first observe, by Corollary 1, that $\inf\{\|x - Tx\| : x \in G\} = 0$. Now, let $\beta_0 = \|z - Tz\| > 0$, and let $0 < \alpha_0 < \beta_0$. We define a sequence $\{\beta_n\}$ by $\beta_n = \xi(\alpha_{n-1})$, where the sequence $\{\alpha_n\}$ is chosen to be positive and satisfies

the following properties: $\alpha_n < \beta_n$ for $n \in \mathbb{N}$ and $\alpha_n \to 0^+$. We now define the set

$$G_n = \{x \in G : \|x - Tx\| < \beta_n\}$$

for $n = 0, 1, 2, \ldots$. Then each G_n is nonempty and open relative to K. Now, without loss of generality, we may assume that these sets are connected, and for each n, we select $x_n \in G_n$. In what follows, we let

$$E_n = \mathrm{co}\{x_k : k \geq n + 1\}$$

where $\mathrm{co}(A)$ denotes the convex hull of A. Following [3], we claim that for $n \leq m$, the set

$$E_{n,m} = \mathrm{co}\{x_k : n + 1 \leq k \leq m + 1\}$$

is contained in G_n for all $n \geq 0$. To this end, we proceed by induction. For $n = m$, the claim clearly holds. We suppose now that $E_{j,m} \subset G_j$ and let $x \in E_{j-1,m}$. Then we have

$$x = \lambda x_j + (1 - \lambda)y$$

for some $y \in E_{j,m} \subset G_j$ and $\lambda \in [0, 1]$. Therefore the Lemma and its proof imply that $x \in G_{j-1}$. Hence $E_{n,m} \subset G_n$ for all $m \geq n \geq 0$, and thus it follows that $E_n \subset G_n$. Since $\overline{E}_{n+1} \subset \overline{E}_n$ and $\overline{E}_n$ is weakly compact, we conclude

$$\varnothing \neq \bigcap_{n=1}^{\infty} \overline{E}_n \subset \bigcap_{n=1}^{\infty} \overline{G}_n.$$

This means that there is $x_0 \in \bigcap_{n=1}^{\infty} \overline{G}_n$, which is a fixed point of T.

We now prove the main result of this paper.

Theorem 1. *Let X be a reflexive Banach space, which satisfies the condition (S), and let D be a bounded open subset of X. Suppose $T : \overline{D} \to X$ is a continuous mapping which is locally pseudo-contractive on D, and suppose there exists $z \in D$ such that*

$$\|z - Tz\| < \|x - Tx\| \quad \textit{for all } x \in \partial D.$$

Then T has a fixed point in D.

Proof. We may assume without loss of generality that D is connected. Also, due to Proposition 2 of [10], the mapping $Fx = 2x - Tx$ is globally one-to-one on D. Therefore, in view of Theorem 2 of [9], we select $w \in D$ so that

$$\|w - Tw\| < \|z - Tz\|.$$

This means that if we define a subset W of D by

$$W = \{x \in D : \|x - Tx\| < \|z - Tz\|\},$$

then $\partial W \subset D$ and

$$\|w - Tw\| < \|x - Tx\| \quad \text{for } x \in \partial W.$$

It follows that $\mathrm{seg}[w, Fw] \subset F(W)$ (see Assertion 1 of [6]), which implies that F^{-1} is nonexpansive on $\mathrm{seg}[w, Fw]$, and thus

$$\|w - F^{-1}(w)\| < \|x - Tx\| \quad \text{for } x \in \partial W.$$

Since $\partial F(W) = F(\partial W)$, we may say that for each $y \in \partial F(W)$, there exists $x \in \partial W$ such that $y = F(x)$ and

$$\|w - F^{-1}(w)\| < \|y - F^{-1}(y)\|.$$

Consequently, by Proposition 2, F^{-1} has a fixed point in D. This completes the proof.

Now, we may derive Theorem 1 of Kirk [6] as a consequence of Theorem 1.

Corollary 2. *Let X be uniformly convex, D a bounded open subset of X, and $T: \overline{D} \to X$ a continuous mapping which is locally pseudo-contractive on D. Suppose there exists $z \in D$ such that*

$$\|z - Tz\| < \|x - Tx\| \quad \text{for all } x \in \partial D.$$

Then T has a fixed point in D.

REFERENCES

1. F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 875–882.

2. _____, *Semicontractive and semiaccretive nonlinear mapping in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.

3. T. H. Chang and C. L. Yen, *Some fixed point theorems in Banach spaces*, J. Math. Anal. Appl. **138** (1989), 550–558.

4. D. Göhde, *Zum frinzip der Kontraktiven Abbildung*, Math. Nachr. **30** (1965), 251–258.

5. T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan **19** (1967), 508–520.

6. W. A. Kirk, *A fixed pint theorem for local pseudo-contractions in uniformly convex spaces*, Manuscripta Math. **30** (1979), 89–102.

7. _____, *Locally nonexpansive mappings in Banach spaces*, Fixed Point Theory (E. Fadell and G. Fournier, eds.), Lecture Notes in Math., vol. 886, Springer-Verlag, Berlin and New York, 1981, pp. 178–198.

8. C. H. Morales, *Pseudo-contractive mappings and the Leray-Schauder boundary condition*, Comment. Math. Univ. Carolin. **20** (1979), 745–756.

9. _____, *On the fixed-point theory for local k-pseudocontractions*, Proc. Amer. Math. Soc. **81** (1981), 71–74.

10. C. H. Morales and S. A. Mutangadura, *On the approximation of fixed points for locally pseudo-contractive mappings*, Proc. Amer. Math. Soc. **123** (1995), 417–423.

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