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ON THE ISOMORPHISM PROBLEM FOR BURNSIDE RINGS

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Dedicated to Professor H.-J. Vollrath on his sixtieth birthday

ABSTRACT. Nonisomorphic 3-groups of distinct nilpotency class are constructed with isomorphic Burnside rings.

1. THE RESULT

W. Burnside in 1911 (see [4, p. 236 ff]) introduced a ring associated to permutation representations of a finite group G which is now called the Burnside ring $\Omega(G)$ of G. Indeed, let $M_1, \dots, M_t$ be a set of representatives of the transitive permutation representations of G. Then every finite G-set M decomposes as a disjoint union of transitive G-sets, so that we can write $M = \lambda_1 M_1 + \cdots + \lambda_t M_t$ for nonnegative integers λ_i. Moreover, if M and N are two G-sets, then there is a natural action of G on the cartesian product $M \times N$ and, by the above, this can be written as an integral linear combination of the M_i. Allowing negative coefficients (i.e. using the usual Grothendieck construction), this yields a ring-structure on the set of all (generalized) permutation representations of G.

The isomorphism problem arises naturally. Indeed, let G and H be finite groups and assume that their Burnside rings $\Omega(G)$ and $\Omega(H)$ are ring-isomorphic. What can be said about G if we know H (see the survey article [8] for the analogous problem for group rings). As the additive group of $\Omega(G)$ is free abelian, freely generated by the representatives of transitive G-sets, we see that the number t of conjugacy classes of subgroups of G and H are equal. Beyond this trivial remark, a celebrated result of A. Dress [5] says that the solubility of H implies solubility of G. Furthermore, a number of other properties can be read off. However, see [9].

The ring $\Omega(G)$ is determined by the table of marks $M(G)$ of G (see [7, Chapter 3]). Thus, for the isomorphism problem for Burnside rings, it is of interest to see what properties can be read off from $M(G)$. Note that the table of marks of G determines the poset $\mathscr{C}(G)$ of conjugacy classes of subgroups of G (see [7, p. 120]). The latter contains much less information about G. Indeed, all groups of order pq have order-isomorphic posets of conjugacy classes. However, it was shown in [1] that $\mathscr{C}(G) \cong \mathscr{C}(H)$ and H a noncyclic p-group

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implies that $|G| = |H|$ and if H is abelian or metacyclic, then G and H are isomorphic (see [1] and [3]). Also, the result proved in [2] can be viewed in this context. Indeed, finite groups were classified with precisely one conjugacy class of nonnormal subgroups, and a trivial consequence of this result is that groups of equal order with this property are isomorphic.

It has been conjectured that $\Omega(G) \cong \Omega(H)$ and H a p-group implies that G and H are of equal nilpotency class. The objective of this note is to construct a counterexample for this. Indeed, we shall prove:

Theorem. *Let* $G = \langle x, y, z | x^9 = y^9 = z^9 = [x, z] = [y, z] = 1, [x, y] = z\rangle$ *and* $H = \langle x, y, z | x^9 = y^9 = z^9 = [y, z] = 1, [x, z] = z^3, [x, y] = z\rangle$. *Then* $\Omega(G) \cong \Omega(H)$. *Moreover,* $|G| = 729 = |H|$ *and* G *is nilpotent of class two and* H *is of class three.*

In particular, the nilpotency class of a p-group cannot in general be read off from its Burnside ring (at least for $p = 3$). It seems likely that the analogous construction works for all primes $p \geq 5$. However, it is not clear how to modify the groups for $p = 2$. Also, it is not known to us whether $c(G) \leq 2$ implies any bound for the class $c(H)$ of H.

2. The isomorphism

First of all, the groups G and H have been shown in [3] to have isomorphic posets of conjugacy classes, so that they were natural candidates to try. The tables of marks for G and H were calculated using the following sequence of GAP commands:

```
x := AbstractGenerator("x");;
y := AbstractGenerator("y");;
z := AbstractGenerator("z");;
G := Group(x, y, z);;
G.relators := [x^9, y^9, z^9, x^-1*z^-1*x*z,
              y^-1*z^-1*y*z, x^-1*y^-1*x*y*z^-1];
p := OperationCosetsFpGroup(G, Subgroup(G, [y]));
pp = TableOfMarks(p);;
LogTo("M(G)");;
DisplayTom(pp);
LogTo ();;
```

and

```
a := AbstractGenerator("a");;
b := AbstractGenerator("b");;
c := AbstractGenerator("c");;
H := Group(a, b, c);;
H.relators := [a^9, b^9, c^9, b^-1*c^-1*b*c,
              a^-1*b^-1*a*b*c^-1, a^-1*c^-1*a*c^-2];
q := OperationCosetsFpGroup(H, Subgroup(H, [b]));;
qq = TableOfMarks(q);;
LogTo("M(H)");;
DisplayTom(qq);
LogTo();;
```

This produced in files $M(G)$ and $M(H)$ two 87×87 lower triangular matrices that describe the multiplication of the corresponding Burnside rings in terms

of the representatives of conjugacy classes of subgroups that will be numbered by $1 \ldots 87$. Instead of the *LogTo* and *DisplayTom* commands we could also use for example

ppp := *MatTom*(*p*); ;
PrintTo("*M*(*G*)", *ppp*); ;

Clearly, a permutation π of the conjugacy classes of subgroups of H does not affect the ring structure of $\Omega(H)$, and our problem was reduced to finding a suitable permutation matrix P related to π such that $P^T M(H) P = M(G)$.

To boil down the possibilities for π for a systematic search, we used the following properties for G and $M(G)$:

(a) Using CAYLEY at Bologna University, we determined that the automorphism group of G is of order $2^4 \cdot 3^9$ and that G has an automorphism φ of order 8. As G is a two generator 3-group, a result of Burnside [4] on coprime automorphisms says that φ acts as an isomorphism of order 8 on the Frattini quotient $G/\Phi(G) \cong Z_3 \otimes Z_3$ of G. Hence $\mathrm{Aut}(G)$ acts transitively on the set of maximal subgroups of G and so without loss of generality, the representative 83 corresponding to one of the maximal subgroups of G may be assumed to be fixed.

(b) If the $n \times n$ matrices $M(G)$ and $M(H)$ satisfy

$$M(G) = P^T M(H) P \tag{1}$$

where P is a permutation matrix, then for all mixed products of the form $q(A, A^T) := A^{k_1}(A^{m_1})^T \cdots A^{k_s}(A^{m_s})^T$ the analogous relation holds

$$q(M(G), M(G)^T) = P^T q(M(H), M(H)^T) P. \tag{2}$$

To determine P with (1) we can consider for example $q_1(A, A^T) := A^3$ or $q_2(A, A^T) := A^2 A^T$ and determine properties that must hold for all permutation matrices (2). A solution P of (1) must satisfy such properties, too. Note that the matrices $q_1(M(G), M(G)^T)$ have more distinct elements with lower frequency. Thus an inspection of $q_1(M(G), M(G)^T)$ provides information about permutations satisfying (1).

(c) For given $n \times n$ matrices A and B we look for a permutation π that satisfies $A = P^T B P$. Each matrix A and every element $s = A_{i,j}$ of A give rise to a graph $G_A(s)$ defined by the knots $P_1, \ldots, P_n$ and the set of edges $\{(P_i, P_j) | \text{ if } A_{i,j} = s\}$. Permutations applied to A permute the knots of $G_A(s)$ in the same way. Hence, $G_A(s)$ and $G_B(s)$ differ only by the permutation π. Comparing these two graphs we can extract information about π. As a trivial example let us consider the case that s appears only once in A and B. Then $G_A(s)$ is given by one edge $\{(P_i, P_j)\}$ and $G_B(s)$ by $\{(P_k, P_m)\}$. Hence, we can deduce that π satisfies $\pi(k) = i$ and $\pi(m) = j$.

Combining (a), (b), and (c) one can determine sufficiently many properties of permutation matrices P that satisfy (1). To compute the matrices $q(A, A^T)$ and the related graphs $G_{q(A,A^T)}(s)$ we used MATLAB. We determined one possible permutation π that transforms the matrix $M(H)$ into the matrix $M(G)$. This permutation π is given by

$$(25, 26)(30, 32)(44, 45)(50, 51)(56, 57)(66, 77, 72)$$
$$(62, 71, 65, 76, 70, 82, 80, 75, 69, 81, 64, 74, 68, 79, 63, 73, 67, 78).$$

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