

ON FUNCTIONS THAT ARE TRIVIAL COCYCLES FOR A SET OF IRRATIONALS. II

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ABSTRACT. Two results are obtained about the topological size of the set of irrationals for which a given function is a trivial cocycle. An example of a continuous function which is a coboundary with non- L^1 cobounding function is constructed.

A function $v : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ is called an (additive) coboundary for an irrational α if there is a measurable function $w : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ such that $v(x) = w(x) - w(x + \alpha)$ a.e. (where we parameterize \mathbb{R}/\mathbb{Z} by the interval $[0,1)$ with addition mod 1). It is called trivial if $v(x) - c$ is a coboundary for some $c \in \mathbb{R}$. In either case the function w , which is unique up to an additive constant, is called the cobounding function. The question of whether particular functions or classes of functions are coboundaries for a given α has applications in ergodic theory and the representation theory of non-Type I groups (see, for example, [BM1],[ILR]). Recent research has revealed an interesting interplay between classes of functions and the types of irrationals for which they can be coboundaries (e.g., [BM2], [M]). Thus it is natural to look at the coboundary question from an opposite point of view, fixing a function v , and asking for exactly which irrationals v is a coboundary. A simple Fourier series argument shows that a trigonometric polynomial must be a coboundary for every irrational α . For other types of functions, this question is much harder to answer.

In a 1988 paper in this journal [B], L. Baggett presented a proof that the set of irrationals for which a given continuous function is a coboundary must be of the first category, unless the function is a trigonometric polynomial. Shortly after this paper appeared, P. Liardet, A. Iwanik, and P. Hellekalek pointed out a gap in the proof. This gap remains unfilled. In this paper, we present an altered version of that proof in the case that the original function is real-analytic. We also present a parallel result for L^1 functions in which the cobounding function is also required to be L^1 . Finally, we display a counterexample which shows that the requirement of an L^1 cobounding function can be a genuine restriction.

Theorem 1. *Let v be an integrable, real-analytic function on the open interval $(0,1)$, which is not a trigonometric polynomial. Then the set S of all irrationals for which v is a trivial cocycle is of the first category.*

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Proof. Suppose S is not of the first category. Let $f(x) = e^{2\pi iv(x)}$. For each integer k and each positive integer j , let $A_{k,j}$ be the set of numbers (rational or irrational) α for which there exists a constant λ and a measurable function g such that

- (1) $\|g\|_2 \leq 1$.
- (2) $|\lambda| = 1$.
- (3) $|\int_0^1 g(x)e^{2\pi ikx} dx| \geq \frac{1}{j}$.
- (4) $f(x)g(x+\alpha) = \lambda g(x)$ for almost all $x \in [0, 1)$.

We claim that each set $A_{k,j}$ is a closed set. Let $\{\alpha_n\}$ be a sequence in $A_{k,j}$ converging to α . For each α_n , there exists a constant λ_n and a measurable function g_n satisfying (1)–(4) above. By passing to a subsequence twice, we may assume that $\lambda_n \rightarrow \lambda$ and $g_n \rightarrow g$ weakly in L^2 . λ and g satisfy conditions (1)–(3) above, and a computation shows that $f(x)g(x+\alpha) = \lambda g(x)$ for almost all $x \in [0, 1)$. Thus $\alpha \in A_{k,j}$.

Clearly $S \subseteq \bigcup_{k,j} A_{k,j}$. By the Baire Theorem, some set A_{k_0,j_0} must contain an open interval. Therefore, there exists a positive integer Q such that for every $q \geq Q$ there is a rational number $p/q \in A_{k_0,j_0}$, and thus a constant λ_q and a function g_q satisfying

- (1) $\|g_q\|_2 \leq 1$.
- (2) $|\lambda_q| = 1$.
- (3) $|\int_0^1 g_q(x)e^{2\pi ik_0 x} dx| \geq \frac{1}{j_0}$.
- (4) $f(x)g_q(x + \frac{p}{q}) = \lambda_q g_q(x)$ for almost all $x \in [0, 1)$.

We also may assume that p is relatively prime to q . It follows by condition (4) that for each $q \geq Q$ there exists a p relatively prime to q such that

$$(*) \quad f(x)f(x + \frac{p}{q})f(x + \frac{2p}{q}) \dots f(x + \frac{(q-1)p}{q}) = \lambda_q^q$$

for every x for which $g_q(x) \neq 0$, and, by condition (3), this is certainly a set of positive measure.

Now the function

$$f(x)f(x + \frac{p}{q}) \dots f(x + \frac{(q-1)p}{q})$$

has discontinuities at most at the multiples of $\frac{p}{q}$, and on each subinterval $(\frac{j}{q}, \frac{(j+1)}{q})$ it is real-analytic. By the identity theorem for real-analytic functions, it follows that

$$f(x)f(x + \frac{p}{q}) \dots f(x + \frac{p(q-1)}{q})$$

is identically λ_q^q on some one of these subintervals. By the invariance of $(*)$ under translation by $\frac{p}{q}$, it follows that

$$f(x)f(x + \frac{p}{q}) \dots f(x + \frac{p(q-1)}{q}) \equiv \lambda_q^q$$

for all x not of the form $\frac{pj}{q}$.

Now $f(x) = e^{2\pi iv(x)}$, so we have that

$$(**) \quad v(x) + v(x + \frac{p}{q}) + \dots + v(x + \frac{p(q-1)}{q}) = c_q + N_q(x)$$

where c_q is a constant and N_q is an integer-valued function. Because v is continuous, we have that N_q is constant on the subintervals $(\frac{j}{q}, \frac{(j+1)}{q})$.

Using (**), we compute the n th Fourier coefficient of v , $c_{nq}(v)$, and obtain

$$qc_{nq}(v) = 0$$

for every nonzero integer n . Since this computation holds for every $q \geq Q$, it follows immediately that v is a trigonometric polynomial. \square

Remark. Michael Herman [H, Theorem 4.11] proved a similar result under the additional hypothesis that for all $n \neq 0$, $c_n(v) \neq 0$.

By requiring the cobounding functions to be integrable, we obtain the following stronger result.

Theorem 2. *Let v be a real-valued L^1 function on \mathbb{R}/\mathbb{Z} , which is not a trigonometric polynomial. Then the set of all irrationals for which v is a trivial cocycle with L^1 cobounding function is of the first category.*

Proof. By the Riemann-Lebesgue Lemma, if v is a coboundary for α with L^1 cobounding function w , then $|c_n(w)| = |c_n(v)|/|1 - e^{2\pi i n \alpha}| \rightarrow 0$ as $|n| \rightarrow \infty$. Thus it will suffice to find a dense G_δ set E of irrationals such that for $\alpha \in E$, $|c_n(v)|/|1 - e^{2\pi i n \alpha}| \not\rightarrow 0$. Since v is not a trigonometric polynomial, $\exists \{m_k\}_{k=1}^\infty$, $m_k \rightarrow \infty$, such that $|c_{m_k}(v)| = \epsilon_k \neq 0$. Choose a_k so that $a_k > \frac{1}{m_k \epsilon_k}$. Let

$$A_k = \bigcup_{j=1}^{m_k-1} \left(\frac{j}{m_k} - \frac{1}{a_k m_k^2}, \frac{j}{m_k} + \frac{1}{a_k m_k^2} \right).$$

If $\alpha \in A_k$, then $\exists j$ such that $\left| \alpha - \frac{j}{m_k} \right| < \frac{1}{a_k m_k^2}$, which implies $|m_k \alpha - j| < \frac{1}{a_k m_k}$ and hence $|1 - e^{2\pi i m_k \alpha}| < \frac{1}{a_k m_k}$. Thus we see that for $\alpha \in A_k$, $|c_{m_k}(v)|/|1 - e^{2\pi i m_k \alpha}| > 1$. Let $E_n = \bigcup_{k=n}^\infty A_k$. E_n is open and dense for each n , and by the above we have that if $\alpha \in E_n$, $\exists m_k$, $k \geq n$, such that $|c_{m_k}(v)|/|1 - e^{2\pi i m_k \alpha}| > 1$. Set $E = \bigcap_{n=1}^\infty E_n$. \square

The apparent advantage of the second theorem over the first raises the natural question of whether an L^1 coboundary, or even an analytic coboundary, must have an L^1 cobounding function.

Theorem 3. *Given any irrational α , there exists a continuous coboundary v for α , whose cobounding function is not L^1 .*

Proof. Choose a sequence of rationals $\{\frac{p_n}{q_n}\}$ satisfying

$$\left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{n^3 2^{2n+1} q_n}.$$

(This can be done by choosing a subsequence of the convergents to α so that each element, $\frac{p_n}{q_n}$, of this subsequence has the property that $q_n \geq n^3 2^{2n+1}$.) For each $n \geq 1$, we define the function u_n by

$$u_n(x) = \begin{cases} 2^{n+1} + n 2^{2n+1} q_n x & \text{if } x \in \left(-\frac{1}{n 2^n q_n}, 0\right), \\ 2^{n+1} - n 2^{2n+1} q_n x & \text{if } x \in \left(0, \frac{1}{n 2^n q_n}\right), \end{cases}$$

and then define

$$w_n(x) = \sum_{p=0}^{q_n-1} u_n\left(x - \frac{p}{q_n}\right).$$

(The function w_n is triangular on $(\frac{p}{q_n} - \frac{1}{n2^n q_n}, \frac{p}{q_n} + \frac{1}{n2^n q_n})$ with $w_n(\frac{p}{q_n}) = 2^{n+1}$ for $p = 0, 1, \dots, q_n - 1$, and 0 everywhere else.) Finally, let

$$w(x) = \sum_{n=1}^{\infty} w_n(x).$$

To show that w is finite a.e., we show that $S_N = \sum_{n=1}^N w_n$ is Cauchy in measure. Indeed, for any $N > M$, $S_N - S_M = \sum_{n=M+1}^N w_n$ is supported on a set of measure $\sum_{n=M+1}^N (q_n)(\frac{1}{n2^n q_n})$, which goes to zero as N and M go to infinity. We see that w is not in L^1 by noting that $\int |w_n(x)| dx = \frac{2}{n}$ so that by the monotone convergence theorem we have

$$\int |w(x)| dx = \sum_{n=1}^{\infty} \int |w_n(x)| dx = \sum_{n=1}^{\infty} \frac{2}{n} = \infty.$$

Now we define

$$v(x) = w(x) - w(x + \alpha) = \sum_{n=1}^{\infty} w_n(x) - w_n(x + \alpha).$$

Since the $w_n(x) - w_n(x + \alpha)$ are continuous, it will follow from the M-test that v is continuous, if we can show that $|w_n(x) - w_n(x + \alpha)| < \frac{1}{n^2}$. By the periodicity of w_n , we have that

$$|w_n(x) - w_n(x + \alpha)| = \left| w_n\left(x + \frac{p_n}{q_n}\right) - w_n(x + \alpha) \right| \leq n2^{2n+1} q_n \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{n^2},$$

since $n2^{2n+1} q_n$ is the maximum slope of a secant line of w_n . \square

Remark. For certain α , we can modify the above construction to give C^r coboundaries with non- L^1 cobounding functions. In particular, if there is a sequence of rational approximations to α , $\{\frac{p_n}{q_n}\}$, such that $|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n^{r+2}}$, we can replace the continuous, piecewise linear functions w_n with C^r , piecewise $(r + 1)$ st degree polynomials, with the same integral as before, and with the property that $\sum w_n^{(r)}(x) - w_n^{(r)}(x + \alpha)$ converges uniformly, thus giving v a continuous r th derivative. Y. Meyer [H, p. 187] has a related result in the $r = 1$ case, which implies that if α has bounded partial quotients in its continued fraction expansion, then there exists a C^1 function which is a coboundary for α with noncontinuous cobounding function. The question of whether there are analytic coboundaries with non- L^1 cobounding functions remains unanswered.

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