

## CONJUGATE BUT NON INNER CONJUGATE SUBFACTORS

MARIE CHODA

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ABSTRACT. It is shown that for each  $r \in \{4 \cos^2(\pi/n) : n \geq 3\} \cup [4, \infty)$ , there exist at least infinitely many subfactors of the hyperfinite  $\text{II}_1$  factor  $R$  with index  $r$ , which are pairwise conjugate but non inner conjugate. In the case that  $r$  is an integer, we have uncountably many such subfactors of  $R$ .

### 1. INTRODUCTION

Two von Neumann subalgebras  $B$  and  $C$  of a von Neumann algebra  $A$  are said to be conjugate if there exists a  $*$ -automorphism  $\theta$  of  $A$  with  $\theta(B) = C$ . If we can take the  $\theta$  as an inner automorphism of  $A$ , then  $B$  and  $C$  are said to be inner conjugate.

Kosaki and Yamagami showed in [12], among other things, that there are infinitely many inner conjugacy classes of subfactors in the hyperfinite  $\text{II}_1$  factor  $R$  with the Jones index 2 ([10]). Otherwise, the subfactors of  $R$  with index 2 are given as the fixed point algebras outer automorphisms on  $R$  with the period 2 ([10]) so that they are all conjugate by [4]. Bearing this fact in mind, we generalize their result in two directions.

One is that, for a given  $r \in \{4 \cos^2(\pi/n) : n \geq 3\} \cup [4, \infty)$ , there exists a countable infinity of non inner conjugate but conjugate subfactors of  $R$  with index  $r$ . More precisely,

**Theorem 1.** *Let  $P$  be a subfactor of the hyperfinite  $\text{II}_1$  factor  $R$  with  $[R : P] < \infty$ . Assume that either  $[R : P] \leq 4$  (except one of type  $A_1^{(1)}$ ) or  $P \subset R$  is Jones' inclusion given in [10]. Then there exists a countable infinity of subfactors of  $R$  which are pairwise non inner conjugate but conjugate to  $P$ .*

A similar result holds for the subfactors of Wenzl ([21]), of Wassermann ([20]) and of Pimsner-Popa ([15, 16]). We construct such subfactors by an easy method (i.e. tensor product). To distinguish the inner conjugacy classes of these subfactors, we use basic (in the sense of [2])  $*$ -endomorphisms of  $R$  instead of bimodules as in [12].

Another generalization is that for each integer  $n$ , there exist uncountably many non inner conjugate subfactors of  $R$  which are pairwise conjugate. More precisely,

**Theorem 2.** *Let  $G$  be a finite group and  $\alpha$  the outer action of  $G$  on the hyperfinite  $\text{II}_1$  factor  $R$ . Then there exist uncountably many subfactors of  $R$  which are pairwise non inner conjugate but conjugate to the fixed point algebra  $R^\alpha$  of  $G$  under  $\alpha$ .*

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2. BASIC \*-ENDOMORPHISMS

Throughout this paper,  $M$  is a  $\text{II}_1$  factor with separable predual and  $\tau$  is the unique faithful normal trace of  $M$ ,  $\tau(1) = 1$ . We denote by  $\text{End}(M, \tau)$  (resp.  $\text{Aut}(M)$ ) the set of trace-preserving \*-endomorphisms (resp. automorphisms) of  $M$ . If  $\sigma \in \text{End}(M, \tau)$ , then  $\sigma$  is unit-preserving, faithful and  $\sigma$ -weakly continuous. Hence  $\sigma(M)$  is a subfactor of  $M$ . Assume that the index  $[M : \sigma(M)] < \infty$ . By iterating the Jones basic construction ([10]) from the inclusion  $\sigma(M) \subset M$ , the tower of  $\text{II}_1$  factors  $\{M_n\}_{n \geq 1}$  is obtained:

$$\sigma(M) \subset M \subset M_1 = \langle M, e_1 \rangle \subset \cdots \subset M_n = \langle M_{n-1}, e_n \rangle \cdots,$$

where  $e_n$  is the Jones projection for the inclusion  $M_{n-2} \subset M_{n-1}$ . The trace  $\tau$  is extended to the unique trace  $\tau$  of  $M_n$  ( $n \geq 1$ ). Let  $M_\infty$  be the  $\text{II}_1$  factor obtained from the GNS construction of  $\bigcup_n M_n$  by  $\tau$ .

A  $\sigma \in \text{End}(M, \tau)$  is called basic ([2]) if  $[M : \sigma(M)] < \infty$  and  $\sigma$  satisfies one of the following equivalent conditions:

- (1)  $\sigma$  is extended to a \*-isomorphism from  $M_1$  onto  $M$ .
- (2) There exists a projection  $e \in M$  such that

$$\sigma^2(M) = \{e\}' \cap \sigma(M), \quad E_{\sigma(M)}(e) = [M : \sigma(M)]^{-1},$$

where  $E_B$  is the  $\tau$ -preserving conditional expectation of  $M$  onto the von Neumann subalgebra  $B$  of  $M$ . Such a projection  $e \in M$  is called a basic projection for  $\sigma$ .

Two \*-endomorphisms  $\rho$  and  $\sigma$  of  $M$  are said to be unitarily equivalent if there exists a unitary  $u \in M$  with  $u\rho(a)u^* = \sigma(a)$  ( $a \in M$ ). The unitary equivalence class of  $\rho$  is denoted by  $[\rho]$ . Then

- (i) if  $[\rho] = [\sigma]$ , then  $[\rho^n] = [\sigma^n]$ , for all  $n$ ;
- (ii) if  $[\rho] = [\sigma \cdot \theta]$  for some  $\theta \in \text{Aut}(M)$ , then  $\rho(A)' \cap A$  is inner conjugate to  $\sigma(A)' \cap A$ .

**Example 1.** Let  $A_0$  be the  $m \times m$  matrices and  $M = \bigotimes_{i=1}^\infty (A_i, \tau_i)$ , where  $A_i = A_0$ , and  $\tau_i$  is the normalized unique trace of  $A_i$  ( $i \geq 1$ ). Let  $\eta_i$  be the canonical embedding of  $A_0$  onto  $A_i$  (under the identification with the subfactor of  $M$ ) with  $\eta_i(x) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots$ . Then the \*-endomorphism  $\sigma$  of  $M$  defined by  $\sigma(\eta_i(x)) = \eta_{i+1}(x)$  ( $x \in A_0$ ) is basic.

**Example 2** (Jones' inclusion). Let  $r \in \{4 \cos^2(\pi/n) : n \geq 3\} \cup [4, \infty)$  and put  $\lambda = r^{-1}$ . Let  $P$  be the factor generated by the sequence of projections  $\{e_i; i \geq 1\}$  with

$$e_i e_j e_i = \begin{cases} \lambda e_i & (|i - j| = 1), \\ e_i e_j & (|i - j| \neq 1), \end{cases}$$

and  $Q = \{e_i : i \geq 2\}''$ . The inclusion  $Q \subset P$  of hyperfinite  $\text{II}_1$  factors is called Jones' inclusion and satisfies that  $[P : Q] = r$ ,  $Q$  is irreducible (i.e.  $Q' \cap P = \mathbf{C}1$ ) if  $r \leq 4$  ([10]) and  $\dim(Q' \cap P) = 2$  if  $r > 4$  ([15]). A basic \*-endomorphism  $\sigma$  of  $P$  with  $\sigma(P) = Q$  is defined by  $\sigma(e_i) = e_{i+1}$ .

**Example 3** (Wenzl's inclusion). The method in Example 2 is generalized by Wenzl ([21]), considering certain unitary representations  $\pi$  of the Braid group on infinitely many generators  $g_0, g_1, \dots$  that admit positive Markov traces. The irreducible inclusion  $N \subset M$  is given as  $N = \{\pi(g_j); j \geq 2\}'' \subset M = \{\pi(g_j); j \geq 1\}''$ . A basic \*-endomorphism  $\sigma$  of  $M$  with  $\sigma(M) = N$  is defined by  $\sigma(\pi(g_i)) = \pi(g_{i+1})$  ( $i \geq 1$ ).

**Example 4** (Ocneanu’s inclusion). The above examples are generalized using bipartite graphs by Ocneanu ([13, 14]). Let  $\mathcal{G}$  be a finite bipartite graph with a biunitary connection  $W$  with respect to the distinguished point  $*$  among the even vertices of  $\mathcal{G}$ . Considering the algebra  $String^{(n)}\mathcal{G}$  of  $n$ -strings on  $\mathcal{G}$ , we give a trace  $\tau$  and an embedding  $i_n^{n+k}$  of  $String^{(n)}\mathcal{G}$  into  $String^{(n+k)}\mathcal{G}$ , which is compatible with  $\tau$ . The trace on  $String^{(n)}\mathcal{G}$  which is compatible with the embedding is unique. The hyperfinite  $\text{II}_1$  factor  $M(\mathcal{G})$  is obtained by the GNS construction of  $(\bigcup_n String^{(n)}\mathcal{G})$  with respect to  $\tau$ . Another embedding  $\varphi$  of  $Strings^{(n)}(\mathcal{G})$  into  $Strings^{(n+1)}(\mathcal{G})$  different with  $i_n^{n+1}$  is induced by the unitary matrix in the biunitary axiom of  $W$ . The  $\varphi$  is a  $*$ -isomorphism compatible with the inclusion  $i_n^{n+k}$  and preserves  $\tau$  so that it is extended to  $M(\mathcal{G})$ . Ocneanu’s subfactor  $N(\mathcal{G})$  is nothing else but  $\varphi(M(\mathcal{G}))$ . The  $\varphi$  is a basic  $*$ -endomorphism of  $M(\mathcal{G})$ .

**Example 5** (Pimsner-Popa’s inclusion). Let  $\sigma$  be the basic  $*$ -endomorphism of  $M$  in Example 1 for  $m = 2$  and  $\eta_i$  be the same as in Example 1. Let  $\theta$  be an outer automorphism of  $M$ . Put

$$\rho(x) = \eta_1(e_{11})\sigma(x) + \eta_1(e_{22})\sigma(\theta(x)) \quad (x \in M),$$

where  $\{e_{ij}\}_{i,j=1,2}$  is the canonical matrix unit of  $M_2(\mathbf{C})$ . Then  $\rho \in \text{End}(M, \tau)$ . The subfactor  $N = \rho(M)$  is Pimsner-Popa’s subfactor ([15, 17]) associated with  $\theta$ . Since  $[\sigma] = [\theta^{-1} \cdot \sigma \cdot \theta]$ ,  $\rho$  is a basic  $*$ -endomorphism of  $M$  with  $\rho(M) = N$ .

**Example 6** (Wasserman’s inclusion). Let  $M$  and  $\sigma$  be the same as in Example 5. Let  $G$  be a subgroup of  $SU(2)$ . Then  $G$  acts on  $M$  by the infinite tensor product of its action by conjugation on  $M_2(\mathbf{C})$ . Let  $A = (M_2(\mathbf{C}) \otimes M)^G \supset B = (1 \otimes M)^G$ . Then  $N$  is a subfactor of  $M$  with index 4 and the principal graph is obtained as one of the extended Coxeter graph corresponding  $G$ ’s ([5, 20]). Since the action of  $G$  commutes with  $\sigma$ ,  $\sigma$  becomes a basic  $*$ -endomorphism of  $M$  with  $\sigma(M) = N$ .

**Proposition 1.** *Let  $N \subset M$  be an inclusion of hyperfinite  $\text{II}_1$  factors with  $[M : N] \leq 4$ . Then there exists a basic  $*$ -endomorphism  $\sigma$  of  $M$  with  $\sigma(M) = N$ .*

*Proof.* The complete classification of subfactors with index  $\leq 4$  of the hyperfinite  $\text{II}_1$  factor is established in [18]. The principal graph of  $N \subset M$  is one of the Coxeter graphs  $A_n$  ( $n \geq 3$ ),  $D_{2n}$  ( $n \geq 2$ ),  $E_6, E_8$  and the extended Coxeter graphs  $A_n^{(1)}$  ( $2 \leq n \leq \infty$ ),  $D_n^{(1)}$  ( $n \geq 4$ ),  $A_\infty, D_\infty, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$  ([1, 5, 6, 7, 8, 10, 11, 13, 14, 16, 17, 19]).

The subfactor of type  $A_n, 3 \leq n < \infty$ , is unique ([10, 11, 13, 14, 16]) and it is a Jones’ subfactor so that a basic  $*$ -endomorphism is given in Example 2. The subfactors of types  $D_{2n}, E_6, E_8, D_n^{(1)}$  are constructed by the method of string algebras via flat connections due to Ocneanu ([7, 8, 11, 13, 14]) and they respectively have basic  $*$ -endomorphisms in Example 4. Each inclusion of type  $A_{2n-1}^{(1)}$  ( $1 \leq n < \infty$ ) is given as Pimsner-Popa’s inclusion  $M \supset N$  associated with the  $\theta = s_n^\gamma$  by Connes ([4]) for some  $\gamma$  with  $\gamma^n = 1$  ([14, 17]). For  $A_\infty^{(1)}$ , the inclusion is unique and the unique (up to the outer conjugacy) aperiodic automorphism ([3]) corresponds to  $\theta$  ([17]). Hence a basic  $*$ -endomorphism of  $M$  of type  $A_{2n-1}^{(1)}$  ( $1 \leq n \leq \infty$ ) with  $N$  as its range is given as  $\rho$  in Example 5. The subfactor with one of  $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$

as the principal graph is unique ([11]), so it is given by Wasserman's method ([5]). Hence it has a basic  $*$ -endomorphism in Example 6.

Thus each subfactor is given as the range of a basic  $*$ -endomorphism.  $\square$

### 3. NON INNER CONJUGATE SUBFACTORS

In this section, we distinguish subfactors of the hyperfinite  $\text{II}_1$  factor which are given through the method of the tensor product by using unitary equivalence relations of basic  $*$ -endomorphisms.

**Lemma 2.** *Let  $\rho, \sigma \in \text{End}(M, \tau)$ . Then  $\rho(M)$  is inner conjugate to  $\sigma(M)$  if and only if there exists a  $\theta \in \text{Aut}(M)$  with*

$$[\sigma \cdot \theta] = [\rho].$$

*Proof.* Assume  $u\rho(M)u^* = \sigma(M)$  for some unitary  $u \in M$ . Then  $\sigma^{-1}(u\rho(M)u^*) = M$ . Putting  $\theta = \sigma^{-1} \cdot \text{Ad}(u) \cdot \rho$ , we have  $\theta \in \text{Aut}(M)$  and  $[\sigma \cdot \theta] = [\rho]$ .  $\square$

**Lemma 3.** *Let  $R$  be the hyperfinite  $\text{II}_1$  factor with the canonical trace  $\tau$  and a basic  $\sigma$  have  $[R : \sigma(R)] = r < \infty$ . Put*

$$M = \bigotimes_{m \in \mathbf{Z}} (R_m, \tau_m), \quad M(i) = \bigotimes_{j < i} (R_j, \tau_j) \otimes (\sigma_i(R), \tau_i) \otimes \bigotimes_{i < k} (R_k, \tau_k),$$

where  $R_m = R, \tau_m = \tau, \sigma_m = \sigma$  ( $m \in \mathbf{Z}$ ). Assume that  $\sigma$  satisfies the condition (\*):

$$(*) \quad \exists n \in \mathbf{N} : \sigma^n(R)' \cap R \text{ is not isomorphic to } (\sigma^{n-1}(R)' \cap R) \otimes (\sigma(R)' \cap R).$$

Then the subfactors  $\{M(i) : i \in \mathbf{Z}\}$  of  $M$  are pairwise conjugate but non inner conjugate.

*Proof.* Let

$$\rho_i = \bigotimes_{j < i} \text{id}_j \otimes \sigma_i \otimes \bigotimes_{i < k} \text{id}_k.$$

Then  $\rho_i$  is a basic  $*$ -endomorphism of  $M$  with  $\rho_i(M) = M(i)$  and  $[M : \rho_i(M)] = [R : P]$ . It is clear that  $\rho_i(M)$  is conjugate to  $\rho_k(M)$  by the automorphism of  $M$  which flips the  $i$ -th factor and the  $k$ -th factor of  $M$ . Assume that  $\rho_i(M)$  is inner conjugate to  $\rho_k(M)$  for  $i \neq k$ . Then there exists a  $\theta \in \text{Aut}(M)$  such that  $[\rho_i \cdot \theta] = [\rho_k]$  by Lemma 2. Hence  $\rho_k^n(M)' \cap M$  is isomorphic to  $\rho_k^{n-1}(\rho_i(M))' \cap M$  by (i) and Lemma 2. On the other hand,  $\rho_k^n(M)' \cap M$  is isomorphic to  $\sigma^n(R)' \cap R$ , but  $\rho_k^{n-1}(\rho_i(M))' \cap M$  is isomorphic to  $(\sigma^{n-1}(R)' \cap R) \otimes (\sigma(R)' \cap R)$ . This contradicts the assumption.  $\square$

*Proof of Theorem 1.* By the assumption, either  $[R : P] \leq 4$  or  $P \subset R$  is Jones' inclusion. Hence by Proposition 1 and Example 2, we have a basic  $\sigma \in \text{End}(R, \tau)$  with  $\sigma(R) = P$ .

For such a pair  $\{R, \sigma\}$ , let  $M$  and  $M(i)$  be the same as in Lemma 3. It is sufficient to prove that the family  $\{M(i)\}_{i \in \mathbf{Z}}$  is pairwise non inner conjugate but  $M(i) \subset M$  is isomorphic to  $P \subset R$ , that is, there is a  $*$ -isomorphism from  $M$  onto  $R$  which transforms  $M(i)$  onto  $P$ .

First we show that  $\sigma$  satisfies the condition (\*) in Lemma 3. Since  $\sigma$  is basic, the  $n$ -th basic extension for  $P \subset R$  is given as  $\sigma^{-n}(R)$ . If the inclusion  $P \subset R$  is irreducible, then (\*) is satisfied by  $n = 2$  because  $\sigma^2(R)' \cap R$  contains a basic projection for  $\sigma$ . Hence (\*) is satisfied if  $P \subset R$  is one of the cases:  $[R : P] < 4$ , or the type of  $A_\infty$  (Jones' inclusion of index 4),  $D_n^{(1)}, D_\infty, E_n^{(1)}$  ( $n = 6, 7, 8$ ). If the principal graph of  $P \subset R$  is  $A_n^{(1)}$  ( $2 \leq n \leq \infty$ ), then  $\sigma(R)' \cap R$  is a 2-dimensional algebra, so that  $(\sigma(R)' \cap R) \otimes (\sigma(R)' \cap R)$  is abelian but  $\sigma^2(R)' \cap R$  is not abelian; thus (\*) is satisfied by  $n = 2$ . The principal graph of Jones' inclusion with index  $> 4$  is also  $A_\infty^{(1)}$  ([5]). Thus (\*) is satisfied by  $n = 2$  for all inclusions. Hence  $M(i)$  is not inner conjugate to  $M(k)$  for  $i \neq k$  by Lemma 3.

Next, we show that  $M(i) \subset M$  is isomorphic to  $P \subset R$ . Remark that the inclusion  $M(i) \subset M$  has the tower of the relative commutants in the basic extensions isomorphic to that of  $P \subset R$ . Popa shows that if  $N \subset M$  is an extremal and strongly amenable inclusion of hyperfinite  $\text{II}_1$  factors, then  $N \subset M$  is opposite isomorphic to  $M'_1 \cap M_\infty \subset M' \cap M_\infty$  ([18, 5.1.2 Corollary]) and that  $N \subset M$  is extremal and strongly amenable if  $[M : N] \leq 4$ . Hence  $M(i) \subset M$  is isomorphic to  $P \subset R$  if  $[R : P] \leq 4$ . If  $P \subset R$  is Jones' inclusion with  $[R : P] > 4$ , then

$$M(i)' \cap M = \mathbf{C}p + \mathbf{C}(1 - p)$$

for a projection  $p \in M$  with  $\tau(p)\tau(1 - p) = [M : M(i)]^{-1}$  and  $\dim(M(i)' \cap M) = 2$ . Hence,  $M(i) \subset M$  is isomorphic to  $P \subset R$  by the characterization of locally trivial subfactors ([18, 5.2.3 Corollary]). Therefore for each given inclusion,  $M(i) \subset M$  is isomorphic to  $P \subset R$ .  $\square$

*Remark.* The proof of Theorem 1 also holds for Wenzl's inclusion and the other inclusion which has the paragroup as the conjugacy invariant.

The most typical subfactor of a finite factor  $M$  with finite index are given as the fixed point algebra  $M^\alpha$  under the outer action  $\alpha$  of a finite group  $G$ . Next we discuss such subfactors.

**Proposition 4.** *Let  $G$  be a finite group and  $\alpha$  an outer action of  $G$  on  $M$ . If  $G$  is nonabelian, then there are no basic \*-endomorphisms of  $M$  which have the subfactor  $M^\alpha$  as the range.*

*Proof.* Assume there exists a basic \*-endomorphism  $\sigma$  of  $M$  with  $\sigma(M) = M^\alpha$ . Let  $M_\infty$  be the enveloping factor obtained from the tower of Jones' basic extension algebra for the inclusion  $M^\alpha \subset M$ . Then we have a  $\beta \in \text{Aut}(M_\infty)$  with  $\beta(M_i) = M_{i-1}$  for all  $i$ , which is the extension of  $\sigma$ . Hence the principal graph and the dual principal graph for  $M^\alpha \subset M$  are isomorphic. This contradicts that  $G$  is nonabelian.  $\square$

By Proposition 4, the proof of Theorem 1 does not work for the inclusion  $M^\alpha \subset M$  to the outer action of a nonabelian finite group  $G$ . In the case of fixed point algebras of finite groups, we use the outerness of actions to distinguish subfactors.

*Proof of Theorem 2.* Let  $I$  and  $J$  be subsets of  $\mathbf{Z}$  with  $I \neq J$ . Let

$$M = \bigotimes_{m \in \mathbf{Z}} (R_m, \tau_m),$$

$$\beta(g) = \bigotimes_{i \in I} \alpha_i(g) \otimes \bigotimes_{i \in \mathbf{Z} \setminus I} \text{id}_i, \quad \gamma(g) = \bigotimes_{i \in J} \alpha_i(g) \otimes \bigotimes_{i \in \mathbf{Z} \setminus J} \text{id}_i.$$

Here  $R_m = R$ ,  $\tau_m = \tau$  (the canonical trace) of  $R$  and  $\alpha_m = \alpha$  ( $m \in \mathbf{Z}$ ).

We denote the canonical trace on  $M$  by the same notation  $\tau$ . Then  $\beta$  and  $\gamma$  are outer actions of  $G$  on  $M$ . Since the outer action of  $G$  on the hyperfinite  $\text{II}_1$  factor is unique up to conjugacy ([9]),  $M^\beta \subset M$  is conjugate to  $R^\alpha \subset R$  for all  $I \subset \mathbf{Z}$ . Assume that  $uM^\beta u^* = M^\gamma$  for some  $u \in M$ . Let  $v$  (resp.  $w$ ) be the unitary representation of  $G$  on  $L^2(M, \tau)$  such that  $\beta(g) = \text{Ad } v(g)$  (resp.  $\gamma(g) = \text{Ad } w(g)$ ). Then  $(uv(G)u^*)' \cap M = u(v(G)' \cap M)u^* = w(G)' \cap M$ . Put  $L = M' \cap B(L^2(M, \tau))$ . Then two crossed products coincide:  $L \triangleleft_{u\beta} G = L \triangleleft_\gamma G$ , where  $(u\beta)(g) = \text{Ad}(uv(g)u^*)$  and  $\gamma(g) = \text{Ad } w(g)$  on  $L$ . Since the actions  $u\beta$  and  $\gamma$  of  $G$  on  $L$  are outer, there exists a conditional expectation  $E$  of  $L \triangleleft_{u\beta} G$  onto  $L$  with  $E(v(g)) = E(uw(g)u^*) = 0$  if  $g \neq 1$ . Let  $uv(g)u^* = \sum_{h \in G} w(h)x(h)$  ( $x(h) \in L$ ) be the Fourier expansion of  $uv(g)u^*$ . Then  $x(1) = 0$ , where  $1$  is the identity of  $G$ . Since  $I \neq J$ ,  $(u\beta)(g^{-1})\gamma(h)$  is an outer automorphism of  $L$  if  $g \neq 1 \neq h$ . Hence we have  $1 = \sum_{h \neq 1} uv(g^{-1})u^*w(h)x(h) = \sum_{h \neq 1} E(uv(g^{-1})u^*w(h)x(h)) = 0$ . This is a contradiction.  $\square$

#### ADDENDUM

A second referee writes that since  $\text{End}_{(M(i))} L^2(M)_{M(j)}$  is not isomorphic to  $\text{End}_{(M(i))} L^2(M)_{M(i)}$ , without any assumptions on values of indices Theorem 1 is proved by Kosaki-Yamagami's bimodule method. Theorem 1 is shown so. However, in the case that  $\sigma(R) \subset R$  is of type  $A_1^{(1)}$ ,  $M(i)$  is inner conjugate to  $M(j)$ .

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DEPARTMENT OF MATHEMATICS, OSAKA KYOIKU UNIVERSITY, ASAHIGAOKA, KASHIWARA 582,  
JAPAN

*E-mail address*: foo527@sinet.ad.jp