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LIMITS OF DIFFERENTIABLE FUNCTIONS

UDAYAN B. DARJI

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ABSTRACT. Suppose that $\{f_n\}$ is a sequence of differentiable functions defined on [0,1] which converges uniformly to some differentiable function f, and $\{f'_n\}$ converges pointwise to some function g. Let $M = \{x : f'(x) \neq g(x)\}$. In this paper we characterize such sets M under various hypotheses. It follows from one of our characterizations that M can be the entire interval [0,1].

1. INTRODUCTION

We say that a sequence of differentiable functions $\{f_n\}$ defined on the interval [0,1] is *proper* if $\{f_n\}$ converges uniformly to some differentiable function f and $\{f'_n\}$ converges pointwise to some function g. For such proper $\{f_n\}$, we let $\Delta(\{f_n\}) = \{x : f'(x) \neq g(x)\}$. It is a standard theorem in elementary analysis texts [6] that if $\{f_n\}$ is proper and $\{f'_n\}$ converges uniformly to some function g, then $\Delta(\{f_n\}) = \emptyset$. It is rather easy to construct an example of a proper $\{f_n\}$ where $\Delta(\{f_n\}) \neq \emptyset$. In this paper we investigate the following questions:

Question 1. *Is there a proper* $\{f_n\}$ *such that* $\Delta(\{f_n\}) = [0,1]$?

Question 2. *Can* $\{\Delta(\{f_n\}) : \{f_n\}$ *is proper* $\}$ *be characterized*?

Theorems 1, 2, 3 and 5 answer Question 2 under various hypotheses. It will follow from Theorem 2 that Question 1 has an affirmative answer. However, Theorem 4 implies that in order to make $\Delta(\{f_n\}) = [0,1]$, the derivatives have to be complicated in some sense.

We now state some definitions and background theorems. Recall that the *density topology* $\mathcal{D}$ on $\mathbb{R}$ is

$$\{M \subset \mathbb{R} : M \text{ is Lebesgue measurable and has density 1 at each of its points}\}.$$

Sets in $\mathcal{D}$ are said to be open in density topology. Whenever we say that a set is open, closed, G_δ, F_σ, etc., we mean that it is open, closed, G_δ, F_σ, etc. in the ordinary topology on $\mathbb{R}$. Whenever we want a set to be open or closed in the density topology, we will specifically so state. A function $f : [0,1] \to \mathbb{R}$ *is approximately*

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continuous iff the preimage of every open set under f is open in the density topology. A set $M \subset [0,1]$ is *nowhere measure dense in the interval* J iff the interior of $M \cap J$ in the density topology is nowhere dense in J in the ordinary topology. We will freely use the following facts about density topology, approximate continuity and Lebesgue integration theory throughout the paper. Their proofs may be found in [1], [3], [2].

Fact 1. *If $M \subset \mathbb{R}$ is measurable, then there exits an F_σ set $N \subset M$ such that N is open in the density topology and $\lambda(M \setminus N)$, the Lebesgue measure of $M \setminus N$, is zero.*

Fact 2. *Every bounded approximately continuous function is a derivative.*

Fact 3 (Zahorski's Theorem [7])**.** *If G_0 and G_1 are two disjoint G_δ sets which are closed in the density topology, then there exists an approximately continuous function $f : [0,1] \to [0,1]$ such that $f^{-1}(i) = G_i$ for $i = 0,1$.*

Fact 4. *If g is an integrable derivative defined on $[0,1]$, then $f(x) = \int_0^x g$ is differentiable everywhere and $f'(x) = g(x)$ for all $x \in [0,1]$.*

Fact 5. *Suppose $\{g_n\}$ is a sequence of integrable functions defined on the interval $[0,1]$ such that $\{g_n\}$ is dominated by an L^1 function and $\{g_n\}$ converges pointwise to g. Let $f_n(x) = \int_0^x g_n$ and $f(x) = \int_0^x g$ for each $x \in [0,1]$. Then $\{f_n\}$ converges uniformly to f.*

2. Main results

Lemma 1. *If $\{f_n\}$ is proper and $\{f'_n\}$ is dominated by an L^1 function, then $\Delta(\{f_n\})$ has measure zero.*

Proof. Without loss of generality assume that $f_n(0) = 0$ for all n. Let g be the pointwise limit of $\{f'_n\}$ and $h(x) = \int_0^x g$. Note that $h'(x) = g(x)$ for almost all $x \in [0,1]$. By the Lebesgue Dominated Convergence Theorem we have that

$$f(x) = \lim_{n\to\infty} f_n(x) = h(x) \quad \text{for all } x \in [0,1].$$

Thus $f' = g$ a.e. and $\Delta(\{f_n\})$ has measure zero. □

Lemma 2. *Suppose $\{f_n\}$ is proper. Then $\Delta(\{f_n\})$ is $G_{\delta\sigma}$.*

Proof. Let g be the pointwise limit of $\{f'_n\}$. Since derivatives are of Baire class 1, g is in Baire class 2 and hence $\Delta(\{f_n\}) = (f' - g)^{-1}(R \setminus \{0\})$ is $G_{\delta\sigma}$ [4]. □

Theorem 1 (General Dominated Case)**.** *A set $M \subset [0,1]$ is $G_{\delta\sigma}$ and of measure zero iff $M = \Delta(\{f_n\})$ for some proper $\{f_n\}$ where $\{f'_n\}$ is dominated by an L^1 function.*

Proof. ($\Leftarrow$) This direction follows from Lemmas 1 and 2.

($\Rightarrow$) The proof of this direction has a flavor similar to a result of Preiss [5]. Let $M = \bigcup G_k$ be a $G_{\delta\sigma}$ set of measure zero where each G_k is G_δ. Now for each positive integer k, let $\{U_{k,n}\}_{n=1}^\infty$ be a decreasing sequence of open sets such that $G_k = \bigcap_{n=1}^\infty U_{k,n}$. For each n and k we may obtain by Fact 3 an approximately continuous function $h_{k,n} : [0,1] \to [0,1]$ such that $h_{k,n}^{-1}(1) = G_k$ and $h_{k,n}^{-1}(0) = (U_{k,n})^c$, the complement of $U_{k,n}$. Note that for each k, $\{h_{k,n}\}_{n=1}^\infty$ converges pointwise to χ_{G_k}, the characteristic function of G_k. Now set

$$g_n = \sum_{k=1}^{\infty} \frac{1}{2^k} h_{k,n} \quad \text{and} \quad g = \sum_{k=1}^{\infty} \frac{1}{2^k} \chi_{G_k}.$$

It follows that g_n is a bounded derivative because it is the uniform sum of a sequence of bounded approximately continuous functions. Also note that $\{g_n\}$ converges pointwise to g.

Now let $f_n(x) = \int_0^x g_n$. That $f_n'(x) = g_n(x)$ for all $x \in [0,1]$ follows from Fact 4. We also have that $\int_0^x g = 0$ for all $x \in [0,1]$ as g is nonzero only on the measure zero set M. Since $\{g_n\}$ is a uniformly bounded sequence of integrable functions which converges pointwise to g, by Fact 5 it follows that $\{f_n\}$ converges uniformly to the zero function. But $\{f_n'\}$ converges pointwise to the function g which is nonzero precisely on set M. Therefore, $\Delta(\{f_n\}) = M$. □

Lemma 3. *Let $M \subset [0,1]$ be a nonempty F_σ set which is open in the density topology, and let $A > 0$. Then, there exists a bounded approximately continuous function $f : [0,1] \to \mathbb{R}$ such that*

(1) $\int_0^1 f = A$,
(2) $f^{-1}(0) = M^c$, *and*
(3) $f \geq 0$.

Proof. By Fact 3 there is a bounded approximately continuous function h which satisifies conditions (2) and (3). Let $f = c \cdot h$, where $c = \frac{A}{\int_0^1 h}$. This f satisfies all three required conditions. □

Lemma 4. *If M is F_σ, open in the density topology and M^c is dense in $[0,1]$, then there exists proper $\{f_n\}$ such that $M = \Delta(\{f_n\})$.*

Proof. Suppose M is F_σ, open in the density topology, and M^c is dense in [0,1]. Let g be an approximately continuous function such that $0 \leq g(x) \leq 1$ for all x and $g^{-1}(0) = M^c$. Write $M = \bigcup_{i=1}^{\infty} F_i$ where each F_i is a nowhere dense closed subset of [0,1]. Since each F_i is a nowhere dense closed subset of [0,1] and M is open in the density topology, it follows that for each interval J for which $M \cap J \neq \emptyset$ we have $(M \cap J) \setminus (\bigcup_{i=1}^{n} F_i) \neq \emptyset$ for all n.

We now construct a sequence of derivatives $\{g_n\}$ in the following fashion. Fix n. Let $N_n = M^c \cup F_1 \cup \ldots \cup F_n$. Observe that N_n is a G_δ set and is also closed in the density topology. Let $\{x(n,k)\}_{k=1}^{m(n)+1}$ be a partition of [0,1] such that $x(n,i) \in M^c$ and $|x(n,i) - x(n,i+1)| < \frac{1}{2^n}$ for each $1 \leq i \leq m(n)$. Fix $1 \leq i \leq m(n)$. Observe that $[x(n,i), x(n,i+1)] \setminus N_n$ is either a nonempty F_σ set which is open in the density topology or $[x(n,i), x(n,i+1)] \subset M^c$. If the latter is the case, let $h_{n,i}$ be the zero function on $[x(n,i), x(n,i+1)]$ and we have that $\int_{x(n,i)}^{x(n,i+1)} h_{n,i} = \int_{x(n,i)}^{x(n,i+1)} g$. Otherwise using Lemma 3, obtain a bounded, nonnegative approximately continuous function $h_{n,i}$ defined on $[x(n,i), x(n,i+1)]$ such that $h_{n,i}^{-1}(0) = N_n \cap [x(i), x(i+1)]$ and $\int_{x(n,i)}^{x(n,i+1)} h_{n,i} = \int_{x(n,i)}^{x(n,i+1)} g$. Now, let h_n be the union of $h_{n,1}, \ldots, h_{n,m(n)}$. Then, h_n is bounded and approximately continuous. Let $g_n = g - h_n$. As $\{h_n\}$ converges pointwise to the zero function, $\{g_n\}$ converges pointwise to g.

Let $f_n(x) = \int_0^x g_n$. Since g_n is bounded and approximately continuous, $f_n' = g_n$. Let us next show that $\|f_n\|$, the sup norm of f_n, is less than 2^{-n+1}. Let $x \in [0,1]$

and i be such that $x \in [x(n,i), x(n,i+1)]$. Then

$$|f_n(x)| = \left|\int_0^x g_n\right| \le \left|\int_0^{x(n,i)} g - h_n\right| + \left|\int_{x(n,i)}^x g - h_n\right|$$
$$\le 0 + \left|\int_{x(n,i)}^x g\right| + \left|\int_{x(n,i)}^x h_n\right| < 2^{-n} + 2^{-n} = 2^{-n+1}.$$

The above estimate on $\|f_n\|$ implies that $\{f_n\}$ converges uniformly to the zero function. We also know that $\{g_n\}$ converges pointwise to g and $g^{-1}(0) = M^c$. Therefore, $M = \Delta(\{f_n\})$. □

Theorem 2 (General Nondominated Case)**.** *A set $M \subset [0,1]$ is $G_{\delta\sigma}$ iff there exists proper $\{f_n\}$ such that $M = \Delta(\{f_n\})$.*

Proof. (⇐) This direction follows from Lemma 2.

(⇒) Let M be $G_{\delta\sigma}$. From Fact 1 obtain two disjoint sets M_1 and M_2 such that $M_1 \cup M_2 = M$, M_1 is $G_{\delta\sigma}$ set of measure zero and M_2 is an F_σ set which is open in the density topology and M_2^c is dense in [0,1]. By Theorem 1 and Lemma 4, obtain proper sequences $\{f_n\}$ and $\{h_n\}$ such that $M_1 = \Delta(\{f_n\})$ and $M_2 = \Delta(\{h_n\})$. Then $\{f_n + h_n\}$ is proper and $M = \Delta(\{f_n + h_n\})$. □

Lemma 5. *Suppose $\{f_n\}$ is proper and for all n, $f_n \in C^1$, i.e. f_n' is continuous. Then $\Delta(\{f_n\})$ is F_σ.*

Proof. Let g be the pointwise limit of $\{f_n'\}$. Then g and f' are of Baire class 1. Therefore, $\Delta(\{f_n\}) = (f' - g)^{-1}(\mathbb{R} \setminus \{0\})$ is F_σ [4]. □

Theorem 3 (Dominated C^1 Case)**.** *A set $M \subset [0,1]$ is F_σ and of measure zero iff $M = \Delta(\{f_n\})$ for some proper $\{f_n\}$ where $f_n \in C^1$ for all n and $\{f_n'\}$ is dominated by an L^1 function.*

Proof. (⇐) This direction follows from Lemmas 1 and 5.

(⇒) Suppose M is F_σ and of measure zero. Let $M = \bigcup_{k=1}^\infty M_k$ where each M_k is closed. Let $\{G_{k,n}\}$ be such that each $G_{k,n}$ is a finite collection of closed intervals and

(1) $\bigcup G_{k,n} \subset \bigcup G_{k,n+1}$, and
(2) $\bigcup_{n=1}^\infty G_{k,n} = M_k^c$.

Now let $h_{k,n}$ be a continuous function defined on [0,1] such that $0 \le h_{k,n}(x) \le 1$, $h_{k,n}(M_k) = 1$, and $h_{k,n}(\bigcup G_{k,n}) = 0$. Note that $\{h_{k,n}\}_{n=1}^\infty$ converges pointwise to χ_{M_k},the characteristic function of M_k. Now set $g_n = \sum_{k=1}^\infty 2^{-k} \cdot h_{k,n}$ and $g = \sum_{k=1}^\infty 2^{-k} \cdot \chi_{M_k}$. Observe that g_n is a continuous function for all n, $0 \le g_n(x) \le 1$ and $\{g_n\}$ converges pointwise to g.

Setting $f_n(x) = \int_0^x g_n$, we have that $f_n'(x) = g_n(x)$ for all $x \in [0,1]$. We also have that $\int_0^x g = 0$ for all $x \in [0,1]$ as g is nonzero only on the measure zero set M. Since $\{g_n\}$ is a uniformly bounded sequence of continuous functions which converges pointwise to g, by Fact 5 it follows that $\{f_n\}$ converges uniformly to the zero function. But $\{f_n'\}$ converges pointwise to function g which is nonzero precisely on set M. Therefore, $\Delta(\{f_n\}) = M$. □

Lemma 6. *Suppose $M \subset [0,1]$ is F_σ and nowhere measure dense. Then M is the union of two disjoint F_σ sets, one of which is of measure zero and the other nowhere dense.*

Proof. Let $B_1, B_2, \ldots$ be a countable basis for [0,1]. Let $U = \bigcup\{B_i : \lambda(B_i \cap M) = 0\}$. Since M is nowhere measure dense, U is a dense open subset of [0,1]. Let $M_1 = M \cap U$ and $M_2 = M \setminus U$. Then M_1 and M_2 are the desired sets. □

Lemma 7. *Let $\{g_n\}$ be a sequence of a.e. continuous functions whose domain is interval I. Suppose $\{g_n\}$ converges pointwise to some function g on I. Then there exists an interval $J \subset I$ and $K > 0$, such that for all n, $|g_n(x)| < K$ for a.e. $x \in J$.*

Proof. We will prove this lemma by contradiction. Assume the hypothesis and that there is no $J \subset I$ such that $\{g_n\}$ is bounded a.e. on J. This implies that for every $K > 0$ and interval $J \subset I$, there are infinitely many integers n such that $|g_n| > K$ on a positive measure set contained in J. Using this observation and the fact that $\{g_n\}$ is a sequence of a.e. continuous functions, we may obtain a subinterval I_1 of I and a positive integer $n(1)$ such that $|g_{n(1)}| > 1$ on I_1. Proceeding in a similar fashion we may obtain a decreasing sequence of closed intervals $\{I_k\}$ and an increasing sequence of positive integers $\{n(k)\}$ such that for each k, $|g_{n(k)}| > k$ on I_k. Now let $p \in \bigcap_{i=1}^{\infty} I_k$. Then $\{g_{n(k)}(p)\}_{k=1}^{\infty}$ does not converge, contradicting the hypothesis and concluding the proof of the lemma. □

Theorem 4. *Suppose $\{f_n\}$ is proper, f_n' is integrable and the set of discontinuities of f_n' has measure zero for all n. Then $\Delta(\{f_n\})$ is nowhere measure dense.*

Proof. Let I be a subinterval of [0,1]. Using Lemma 7 obtain an interval $J \subset I$ such that $\{f_n'\}$ is bounded a.e. on J. By Lemma 1 we have that $\Delta(\{f_n\}) \cap J$ has measure zero. This implies that the interior of $\Delta(\{f_n\})$ in the density topology is nowhere dense in [0,1]. □

Lemma 8. *Suppose M is an F_σ nowhere dense subset of $[0,1]$. Then $M = \Delta(\{f_n\})$ for some proper $\{f_n\}$ where $f_n \in C^1$ for all n.*

Proof. Let $U = (\mathrm{cl}(M))^c$ and $F_1, F_2, \ldots$ be a pairwise disjoint decomposition of M into closed sets. Let g be the function which is zero on M^c and 2^{-i} on F_i. Note that g is of Baire class 1. Let $\{G_n\}$ be such that each G_n is a finite collection of closed intervals and

- $\bigcup G_n \subset \bigcup G_{n+1}$, and
- $\bigcup_{n=1}^{\infty}(\bigcup G_n) = U$.

Using the fact that g is of Baire class 1, obtain a sequence of continuous functions $\{h_n\}$ such that $\{h_n\}$ converges pointwise to g, and for all n, $h_n(\bigcup G_n)) = 0$ and $0 \le h_n(x) \le 1$.

We now construct g_n in the following manner. First, let $\{x(n,k)\}_{k=1}^{m(n)+1}$ be a partition of [0,1] such that $|x(n,k) - x(n,k+1)| < 2^{-n}$ for $k = 1, 2, \ldots, m(n)$. Now for each $1 \le k \le m(n)$, let $a_{n,k}$ be a continuous nonnegative function defined on $[x(n,k), x(n,k+1)]$ such that

- $\int_{x(n,k)}^{x(n,k+1)} a_{n,k} = \int_{x(n,k)}^{x(n,k+1)} h_n$,
- $a_{n,k} = 0$ on $\bigcup G_n \cap [x(n,k), x(n,k+1)]$,
- $a_{n,k}^{-1}(\mathbb{R} \setminus \{0\}) \subset U$, and
- $a_{n,k}(x(n,k)) = a_{n,k}(x(n,k+1)) = 0$.

Let a_n be the union of $a_{n,1}, a_{n,2}, \dots, a_{n,m(n)}$ and $g_n = h_n - a_n$. As $\{a_n\}$ is a sequence of continuous functions which converges pointwise to the zero function, $\{g_n\}$ is a sequence of continuous functions which converges pointwise to g.

For each n, let $f_n(x) = \int_0^x g_n$. Observe that for $1 \le k \le m(n)+1$

$$f_n(x(n,k)) = \int_0^{x(n,k)} g_n = \sum_{i=1}^{k-1} \int_{x(n,i)}^{x(n,i+1)} h_n - a_n = 0.$$

Using this observation we obtain an estimate on $\|f_n\|$. Let $x \in [0,1]$. Let k be a positive integer such that $x \in [x(n,k), x(n,k+1)]$.

$$\begin{aligned} |f_n(x)| &\le |f_n(x(n,k))| + |f_n(x) - f_n(x(n,k))| \\ &= 0 + \left| \int_{x(n,k)}^{x} g_n \right| = \left| \int_{x(n,k)}^{x} h_n - a_n \right| \\ &\le \left| \int_{x(n,k)}^{x} h_n \right| + \left| \int_{x(n,k)}^{x} a_n \right| \\ &< 2^{-n} + 2^{-n} = 2^{-n+1}. \end{aligned}$$

From above we have that $\{f_n\}$ converges uniformly to the zero function. As $\{f_n'\}$ converges pointwise to g and $g^{-1}(0) = M^c$, we have that $M = \Delta(\{f_n\})$. □

Theorem 5 (Nondominated C^1 Case). *A set $M \subset [0,1]$ is F_σ and nowhere measure dense iff $M = \Delta(\{f_n\})$ for some proper $\{f_n\}$ where $f_n \in C^1$ for all n.*

Proof. (⇐) This direction follows from Theorem 4 and Lemma 5.

(⇒) Suppose M is F_σ and nowhere measure dense. Then by Lemma 6, $M = M_1 \cup M_2$ where M_1 and M_2 are disjoint F_σ sets, one of which is nowhere dense and the other of measure zero. By Theorem 3 and Lemma 8, there are proper sequences $\{f_n\}$ and $\{h_n\}$ such that $\Delta(\{f_n\}) = M_1$ and $\Delta(\{h_n\}) = M_2$. Then $\{f_n + h_n\}$ is proper and $M = \Delta(\{f_n + h_n\})$. □

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Department of Mathematics, University of Louisville, Louisville, Kentucky 40292
E-mail address: ubdarj01@homer.louisville.edu