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A RIGIDITY THEOREM FOR THE CLIFFORD TORI IN S^3

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ABSTRACT. Let S^3 be the unit hypersphere in the 4-dimensional Euclidean space $\mathbb{R}^4$ defined by $\sum_{i=1}^4 x_i^2 = 1$. For each θ with $0 < \theta < \pi/2$, we denote by M_θ the Clifford torus in S^3 given by the equations $x_1^2 + x_2^2 = \cos^2\theta$ and $x_3^2 + x_4^2 = \sin^2\theta$. The Clifford torus M_θ is a flat Riemannian manifold equipped with the metric induced by the inclusion map $i_\theta\colon M_\theta \to S^3$. In this note we prove the following rigidity theorem: If $f\colon M_\theta \to S^3$ is an isometric embedding, then there exists an isometry A of S^3 such that $f = A \circ i_\theta$. We also show no flat torus with the intrinsic diameter $\leq \pi$ is embeddable in S^3 except for a Clifford torus.

1. Introduction

Let S^3 be the unit hypersphere in the 4-dimensional Euclidean space $\mathbb{R}^4$ defined by $\sum_{i=1}^4 x_i^2 = 1$. For each θ with $0 < \theta < \pi/2$, we denote by M_θ the Clifford torus in S^3 given by

$$x_1^2 + x_2^2 = \cos^2\theta, \qquad x_3^2 + x_4^2 = \sin^2\theta.$$

The Clifford torus M_θ is a flat Riemannian manifold equipped with the metric induced by the inclusion map $i_\theta\colon M_\theta \to S^3$. The authors are interested in the following question: For every isometric immersion $f\colon M_\theta \to S^3$, does there exist an isometry A of S^3 such that $f = A \circ i_\theta$? Concerning this question, it is known that if $f_t\colon M_\theta \to S^3$, $-\infty < t < \infty$, is a smooth 1-parameter family of isometric immersions with $f_0 = i_\theta$, then for each t there exists an isometry A_t of S^3 such that $f_t = A_t \circ i_\theta$. However, the question above seems not to have been settled yet. In this note we give an affirmative answer to the question under the assumption that the immersion f is an embedding. In other words, we prove the following rigidity theorem.

Theorem 1. *If $f\colon M_\theta \to S^3$ is an isometric embedding, then there exists an isometry A of S^3 such that $f = A \circ i_\theta$.*

For each isometric immersion $f\colon M_\theta \to S^3$, we denote by $\mathrm{Diam}(f)$ the diameter of the image $f(M_\theta)$ in S^3. Note that $\mathrm{Diam}(i_\theta) = \pi$. The following theorem, which will be proved in §2, is a key ingredient in the proof of Theorem 1.

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Theorem 2. *If* $f\colon M_\theta \to S^3$ *is an isometric immersion with* $\mathrm{Diam}(f) = \pi$*, then there exists an isometry* A *of* S^3 *such that* $f = A \circ i_\theta$*.*

We now give the proof of Theorem 1. It follows from [2] that if f is an isometric embedding of a flat torus M into S^3, then the image $f(M)$ is invariant under the antipodal map of S^3. In particular, for each isometric embedding $f\colon M_\theta \to S^3$, we have $\mathrm{Diam}(f) = \pi$. Therefore the assertion of Theorem 1 follows from Theorem 2.

In §3, we obtain Theorem 3, which generalizes Theorem 2 for an isometric immersion of a flat torus with intrinsic diameter less than or equal to π. An immediate consequence of Theorem 3 is that the only flat tori with intrinsic diameter $\leq \pi$ which can be embedded in S^3 are Clifford tori.

2. Proof of Theorem 2

We define a Riemannian covering map $T\colon \mathbb{R}^2 \to M_\theta$ of the 2-dimensional Euclidean space $\mathbb{R}^2$ into the Clifford torus M_θ by setting

$$T(u_1, u_2) = \left(R_1 \cos\left(\frac{u_1}{R_1}\right), R_1 \sin\left(\frac{u_1}{R_1}\right), R_2 \cos\left(\frac{u_2}{R_2}\right), R_2 \sin\left(\frac{u_2}{R_2}\right)\right),$$

where $R_1 = \cos\theta$ and $R_2 = \sin\theta$. Note that $T(u_1, u_2) = T(u_1 + l_1, u_2 + l_2)$ if and only if $l_i/2\pi R_i$ is an integer for each i. Let V_1 and V_2 be the vector fields on M_θ given by

$$\begin{cases} V_1(T(u_1, u_2)) = \frac{d}{dt}\Big|_{t=0} T(u_1 + R_1 t, u_2 + R_2 t), \\ V_2(T(u_1, u_2)) = \frac{d}{dt}\Big|_{t=0} T(u_1 + R_1 t, u_2 - R_2 t). \end{cases} \tag{1}$$

Then we have

$$g(V_1, V_1) = g(V_2, V_2) = 1, \qquad g(V_1, V_2) = \cos 2\theta, \tag{2}$$

where g denotes the Riemannian metric on M_θ. For $i = 1, 2$, we denote by $\{\varphi_i^t\}$ the 1-parameter group of transformations of M_θ generated by the vector field V_i.

Lemma 1. *Let* $f\colon M_\theta \to S^3$ *be an isometric immersion, and let* p *be a point in* M_θ*. If there exists a point* $q \in M_\theta$ *such that* $f(p) = -f(q)$*, then the curve* $\gamma_i(t) = f(\varphi_i^t(p))$ *is a unit speed geodesic in* S^3*.*

Proof. Take a point $(a_1, a_2) \in \mathbb{R}^2$ such that $T(a_1, a_2) = p$. By (1) we obtain

$$\begin{cases} \varphi_1^t(p) = T(a_1 + R_1 t, a_2 + R_2 t), \\ \varphi_2^t(p) = T(a_1 + R_1 t, a_2 - R_2 t). \end{cases} \tag{3}$$

Let $d(p, q)$ denote the intrinsic distance between p and q in M_θ. Then it follows from $f(p) = -f(q)$ that $d(p, q) \geq \pi$. Since the intrinsic diameter of M_θ is equal to π, we obtain $d(p, q) = \pi$. Hence

$$q = T(a_1 + R_1\pi, a_2 + R_2\pi) = T(a_1 + R_1\pi, a_2 - R_2\pi). \tag{4}$$

It follows from (3) and (4) that $\gamma_i(t)$ is a unit speed curve in S^3 such that $\gamma_i(0) = \gamma_i(2\pi) = f(p)$ and $\gamma_i(\pi) = f(q) = -f(p)$. This shows that $\gamma_i|[0, 2\pi]$ is a geodesic in S^3. Since $\gamma_i(t + 2\pi) = \gamma_i(t)$, the curve $\gamma_i(t)$ is a unit speed geodesic in S^3. □

Lemma 2. *Let* $f\colon M_\theta \to S^3$ *be an isometric immersion with* $\mathrm{Diam}(f) = \pi$*, and let* h *denote the second fundamental form of the immersion* f*. Then* $h(V_1, V_1) = h(V_2, V_2) = 0$ *and* $|h(V_1, V_2)| = \sin 2\theta$*.*

Proof. We set $h_{ij} = h(V_i, V_j)$. By (2) and the equation of Gauss, we obtain

$$\langle h_{12}, h_{12}\rangle - \langle h_{11}, h_{22}\rangle = \sin^2 2\theta,$$

where $\langle\ ,\ \rangle$ denotes the Riemannian metric on S^3. We now define M_θ^* to be the set of all $p \in M_\theta$ such that $f(p) = -f(q)$ for some $q \in M_\theta$. Using Lemma 1, we see that $h_{11} = h_{22} = 0$ on M_θ^*. So it is sufficient to show that $M_\theta^* = M_\theta$. Since $\mathrm{Diam}(f) = \pi$, there exists a point $p \in M_\theta^*$. Let $c(s)$ be the curve in M_θ given by $c(s) = \varphi_1^s(p)$. Then it follows from Lemma 1 that the curve $f(c(s))$ is a unit speed geodesic in S^3, and so $f(c(s)) = -f(c(s+\pi))$. Hence $c(s) \in M_\theta^*$ for all s. For each $s \in \mathbb{R}$, let $c_s(t)$ be the curve in M_θ given by $c_s(t) = \varphi_2^t(c(s))$. By the same way as above we see that $c_s(t) \in M_\theta^*$. Hence $\varphi_2^t(\varphi_1^s(p)) \in M_\theta^*$ for all $(s,t) \in \mathbb{R}^2$. This implies $M_\theta^* = M_\theta$. □

We now give the proof of Theorem 2. Let $f\colon M_\theta \to S^3$ be an isometric immersion with $\mathrm{Diam}(f) = \pi$. We set $f_1 = i_\theta$ and $f_2 = f$. For $k = 1, 2$, let h_k be the second fundamental form of the immersion f_k, and let $\xi_k = h_k(V_1, V_2)/\sin 2\theta$. Then it follows from Lemma 2 that ξ_k defines a unit normal vector field along f_k, and

$$\langle h_1(V_i, V_j), \xi_1\rangle = \langle h_2(V_i, V_j), \xi_2\rangle.$$

Hence the fundamental theorem of the theory of surfaces implies that there exists an isometry A of S^3 such that $f_2 = A \circ f_1$. This completes the proof of Theorem 2.

3. A generalization of Theorem 2

In this section we generalize Theorem 2 as follows.

Theorem 3. *Let M be a flat torus with intrinsic diameter less than or equal to π. If $f\colon M \to S^3$ is an isometric immersion with* $\mathrm{Diam}(f) = \pi$*, then there exist an isometry $\varphi\colon M_\theta \to M$ for some $\theta \in (0, \pi/2)$ and an isometry A of S^3 such that $f \circ \varphi = A \circ i_\theta$.*

Proof. Since $\mathrm{Diam}(f) = \pi$, there exist points p, q in M such that $f(p) = -f(q)$. Let $d(p,q)$ denote the intrinsic distance between p and q in M. It follows from $f(p) = -f(q)$ that $d(p,q) \geq \pi$. But our assumption on the intrinsic diameter of M implies that $d(p,q) = \pi$. In addition, any geodesic of M of length π which connects p and q is mapped by f to a geodesic S^3 which connects $f(p)$ to $f(q)$.

If M is not isometric to M_θ, we claim that p and q are connected by three geodesics in M of length π whose tangent vectors at p lie in three distinct linear subspaces. To prove the claim consider the Riemannian covering map $k\colon \mathbb{R}^2 \to M$. We suppose that 0, the origin, lies in $\Gamma = k^{-1}(q)$; of course, M is isometric to $\mathbb{R}^2/\Gamma$. Let q_1 and q_2 be the elements of smallest norm in $\Gamma\backslash\{0\}$ and $\Gamma\backslash\{nq_1 : n \in \mathbb{Z}\}$, respectively. For notational reasons, let $0 = q_0$. Denote the triangle with vertices q_i, for $i = 0, 1, 2$, by Δ. One may show that the circumcenter of Δ, denote p_0, lies in $k^{-1}(p)$ and $d(p_0, q_i) = \pi$, for $i = 0, 1, 2$. Thus there are at least three geodesics in M of length π connecting p to q. Assume that the tangent vectors to these geodesic arcs at p must lie in a pair of linear subspaces of the tangent space to M at p. Necessarily, two of the segments $\overline{p_0 q_i}$ lie on the same line. Say, for example, that the segment $\overline{q_1 q_2} = \overline{q_1 p_0} \cup \overline{p_0 q_2}$. Since the circumcenter of Δ is on the side $\overline{q_1 q_2}$ of Δ, one sees that Δ is a right triangle with right angle at q_0. It follows that M is isometric to a Clifford torus. This contradiction proves the claim.

If M is not isometric to M_θ, then the images under f of the geodesic segments mentioned in the previous paragraph are geodesics of S^3. This implies that the immersion f is totally geodesic at p, which is impossible. Hence M must be intrinsically isometric to M_θ. Now Theorem 3 follows from Theorem 2. □

Theorem 4. *It is impossible to embed a flat torus with intrinsic diameter* $\leq \pi$ *in* S^3 *unless the flat torus is a Clifford torus.*

Proof. Again, from [2], the image of any embedding must have antipodal symmetry. Thus the assertion of this theorem follows from Theorem 3. □

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