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A PROPORTIONAL DVORETZKY-ROGERS FACTORIZATION RESULT

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Abstract. If X is an n-dimensional normed space and $\varepsilon \in (0,1)$, there exists $m \geq (1-\varepsilon)n$, such that the formal identity $i_{2,\infty}\colon l_2^m \to l_\infty^m$ can be written as $i_{2,\infty} = \alpha \circ \beta, \beta\colon l_2^m \to X, \alpha\colon X \to l_\infty^m$, with $\|\alpha\| \cdot \|\beta\| \leq c/\varepsilon$. This is proved as a consequence of a Sauer-Shelah type theorem for ellipsoids.

1. Introduction

A version of the classical Dvoretzky-Rogers lemma [D-R] asserts that, if $(X, \|\cdot\|)$ is an n-dimensional normed space, there exist vectors $x_1, \ldots, x_m \in X$, $m = [\sqrt{n}]$, such that for any choice of real numbers $t_1, \ldots, t_m$,

$$\max_{j \leq m} |t_j| \leq \left\| \sum_{j \leq m} t_j x_j \right\|_X \leq c \left(\sum_{j \leq m} t_j^2 \right)^{1/2},$$

where $c > 0$ is an absolute constant. Towards a strengthening of this result for m proportional to n, Bourgain-Szarek [B-S] and later Szarek-Talagrand [S-T] proved the following:

Theorem 1. *If $(X, \|\cdot\|)$ is an n-dimensional normed space and $\varepsilon \in (0,1)$, there exist vectors $x_1, \ldots, x_m \in X$, $m \geq (1-\varepsilon)n$, such that for any reals $t_1, \ldots, t_m$,*

$$\max_{j \leq m} |t_j| \leq \left\| \sum_{j \leq m} t_j x_j \right\|_X \leq c\varepsilon^{-d} \left(\sum_{j \leq m} t_j^2 \right)^{1/2},$$

where $c, d > 0$ are absolute constants. Equivalently, the formal identity $i_{2,\infty} : l_2^m \to l_\infty^m$ can be written as $i_{2,\infty} = \alpha \circ \beta$, where $\beta\colon l_2^m \to X, \alpha\colon X \to l_\infty^m$, and $\|\alpha\| \cdot \|\beta\| \leq c\varepsilon^{-d}$. The same holds true for $i_{1,2}\colon l_1^m \to l_2^m$.

The best possible dependence on ε is not known. As shown by S. J. Szarek [Sz.1], there exists an n-dimensional normed space X such that $\|\alpha\| \cdot \|\beta\| \geq c(n/\log n)^{1/10}$ whenever $i_{2,\infty}\colon l_2^n \to l_\infty^n$ is written as $i_{2,\infty} = \alpha \circ \beta$ (α, β as above), and this implies that d in Theorem 1 has to be at least $1/10$. On the other hand, in [S-T] it is proved that Theorem 1 holds with $d = 2$, and in [G] we obtain a similar result with $d = 3/2$. Here, we shall show that the same holds true with $d = 1$.

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Let us note that the method establishing this "proportional Dvoretzky-Rogers factorization" is closely related to the problem of the Banach-Mazur distance to the cube. A detailed exposition of the techniques used so far for both problems is given in [Sz.2].

The source of the improvement on the estimates in Theorem 1 is a Sauer-Shelah type theorem for ellipsoids, which we feel is of independent interest: The well-known combinatorial Sauer-Shelah lemma [Sa], [Sh] states that if $0 \leq l < s$ and M is a subset of $\{-1,1\}^s$ of cardinality $|M| > \binom{s}{0} + \binom{s}{1} + \cdots + \binom{s}{l}$, then there exists $\sigma \subset \{1,\ldots,s\}$, $|\sigma| > l$, such that $P_\sigma(M) = \{-1,1\}^\sigma$, where P_σ is the restriction map $(\delta_j)_{j\leq s} \to (\delta_j)_{j\in\sigma}$. A special case of this lemma is of particular interest: If $M \subset \{-1,1\}^s$ and $|M| \geq 2^{s-1}$, then we can find $\sigma \subset \{1,\ldots,s\}, |\sigma| \geq \frac{s}{2}$, with $P_\sigma(M) = \{-1,1\}^\sigma$.

In connection with their work on the Banach-Mazur distance to the cube, Szarek and Talagrand [S-T] proved an isomorphic variant of the Sauer-Shelah lemma: If $M \subset \{-1,1\}^s$, viewed now as a set of points in $\mathbf{R}^s$, and if $|M| \geq 2^{s-1}, \varepsilon \in (0,1)$, then there exists $\sigma \subset \{1,\ldots,s\}$, $|\sigma| \geq (1-\varepsilon)s$, such that

$$\mathrm{absconv}(P_\sigma(M)) \supseteq c\varepsilon[-1,1]^\sigma,$$

where $c > 0$ is an absolute constant (and the absolute convex hull is taken in $\mathbf{R}^\sigma$).

For our purposes, we need to consider the following situation: Let $u_1,\ldots,u_s$ be vectors in $\mathbf{R}^n$, of Euclidean norm $|u_j|_n \leq 1$, $j = 1,\ldots,s$. Define the symmetric convex set

$$\mathcal{E} = \left\{ (\delta_j)_{j\leq s} \in \mathbf{R}^s \colon \left| \sum_{j\leq s} \delta_j u_j \right|_n \leq 1 \right\}.$$

(Note that if $s \leq n$ and the vectors u_j are linearly independent in $\mathbf{R}^n$, then $\mathcal{E}$ is an ellipsoid in $\mathbf{R}^s$. This will be the context in the proof of Theorem 1.) Again, we are interested in the "size" of the image $P_\sigma(\mathcal{E})$ of $\mathcal{E}$ for "large" subsets σ of $\{1,\ldots,s\}$. Our main result is then the following

Theorem 2. *If $u_j \in \mathbf{R}^n, |u_j|_n \leq 1$, $j = 1,\ldots,s$, and*

$$\mathcal{E} = \left\{ (\delta_j)_{j\leq s} \in \mathbf{R}^s \colon \left| \sum_{j\leq s} \delta_j u_j \right|_n \leq 1 \right\},$$

then for every $\varepsilon \in (0,1)$, we can find $\sigma \subseteq \{1,\ldots,s\}$, $|\sigma| \geq (1-\varepsilon)s$, such that

$$P_\sigma(\mathcal{E}) \supseteq c\sqrt{\varepsilon} D_\sigma,$$

where D_σ is the Euclidean unit ball in $\mathbf{R}^\sigma$ and $c > 0$ is an absolute constant.

We shall use the standard notation from [M-Sc] or [T-J]. By $|\cdot|$ we denote the cardinality of a finite set. The letter c will always denote an absolute positive constant, not necessarily the same in all its occurrences. For basic facts about p-absolutely summing operators, used in the proof of Theorem 2, we refer the reader to [L-T], [Pi], and [T-J].

2. Proof of Theorem 2

First, we introduce some additional notation: The set $S = \{1,\ldots,s\}$, as well as $\mathbf{R}^s$, will be fixed throughout the proof. If $\varphi \subseteq S$, then $\mathbf{R}^\varphi = \{(\delta_j)_{j\leq s} \in \mathbf{R}^s \colon \delta_j = 0$ if $j \notin \varphi\}$. A point in $\mathbf{R}^s$ denoted by $(\delta_j)_{j\in\varphi}$ is assumed to satisfy $\delta_j = 0$ if $j \notin \varphi$.

If τ, φ are disjoint subsets of S and $A \subseteq \mathbf{R}^{\varphi}$, we sometimes write $\mathbf{0}_{\tau} \times A$ instead of A to indicate that A is to be understood as a subset of $\mathbf{R}^{\varphi \cup \tau}$. In particular, if $\varphi \subseteq S_1 \subseteq S$ and $L > 0$, then

$$\begin{aligned} I_{L,\varphi,S_1} &= \mathbf{0}_{\varphi} \times \{-L, L\}^{S_1 \backslash \varphi} \\ &= \left\{ (\delta_j)_{j \in S_1} \in \mathbf{R}^{S_1} \colon \delta_j = \begin{cases} 0 & \text{if } j \in \varphi, \\ \pm L & \text{if } j \in S_1 \backslash \varphi \end{cases} \right\}. \end{aligned}$$

Note that $|I_{L,\varphi,S_1}| = 2^{|S_1 \backslash \varphi|}$. If $a \in \mathbf{R}^{\varphi}$, $b \in \mathbf{R}^{\tau}$, and τ, φ are disjoint subsets of S, then $(a, b) \in \mathbf{R}^{\varphi \cup \tau}$ is the sum $a + b$. Finally, if S_1 is a non-empty subset of S, we define

$$\mathcal{E}_{S_1} = \left\{ (\delta_j)_{j \in S_1} \in \mathbf{R}^{S_1} \colon \left| \sum_{j \in S_1} \delta_j u_j \right|_n \leq 1 \right\}.$$

Our starting point is then an immediate consequence of the Sauer-Shelah lemma:

Lemma 1. *If $L > 0$, $\varphi \subseteq S_1 \subseteq S$, and $M \subseteq \mathbf{0}_{\varphi} \times \{-L, L\}^{S_1 \backslash \varphi}$, with $|M| \geq 2^{|S_1 \backslash \varphi| - 1}$, then there exists $\sigma \subseteq S_1 \backslash \varphi, |\sigma| \geq \frac{|S_1 \backslash \varphi|}{2}$, such that*

$$P_{\varphi \cup \sigma}(M) = \mathbf{0}_{\varphi} \times \{-L, L\}^{\sigma}. \quad \square$$

Using an inductive argument based on Lemma 1, we obtain a first result on the size of the projections of $\mathcal{E}_{S_1}$, for an arbitrary $S_1 \subseteq S$. This step is crucial for our proof of Theorem 2, so we state it as our next lemma and give its proof, although it can essentially be found in [G].

Lemma 2. *If $\emptyset \neq S_1 \subseteq S$ and $\varepsilon \in (0,1)$ are given, then there exists $\sigma \subseteq S_1$, with $|\sigma| \geq (1 - \varepsilon)|S_1|$, such that*

$$P_{\sigma}(\mathcal{E}_{S_1}) \supseteq \frac{c\sqrt{\varepsilon}}{\sqrt{|S_1|}}[-1,1]^{\sigma},$$

where $c > 0$ is an absolute constant.

Proof. Set $\alpha_k = \sum_{r=0}^{k-1} 2^{r/2}$, $\beta_k = \sum_{r=0}^{k-1} 2^r = 2^k - 1$, and $Q_{\tau} = [-1,1]^{\tau}$ for every non-empty $\tau \subseteq S_1$.

We shall prove by induction that:

$$(*) \qquad \begin{aligned} &\textit{For } k = 1, 2, \ldots, \textit{ one can find } \sigma_k \subseteq S_1, \textit{ with } |\sigma_k| \geq \\ &(1 - \tfrac{1}{2^k})|S_1|, \textit{such that} \\ &Q_{\sigma_k} \subseteq P_{\sigma_k}(\alpha_k \sqrt{2|S_1|}\mathcal{E}_{S_1} \cap \beta_k Q_{S_1}). \end{aligned}$$

Since $\alpha_k \leq \frac{2^{k/2}}{\sqrt{2}-1}$, condition $(*)$ clearly implies that, for $k = 1, 2, \ldots$,

$$P_{\sigma_k}(\mathcal{E}_{S_1}) \supseteq \frac{c}{\sqrt{|S_1|}} \sqrt{\frac{1}{2^k}} [-1,1]^{\sigma_k}$$

with $c = 1 - \frac{1}{\sqrt{2}}$, which is the assertion of the lemma for $\varepsilon = 1/2^k$. The continuous version will easily follow with a worse constant c.

Inductive step. Consider the set $J_k = \mathbf{0}_{\sigma_k} \times \{-2^{k/2}, 2^{k/2}\}^{S_1 \backslash \sigma_k}$, where σ_k is the subset of S_1 given by $(*)$. Note that $|J_k| = 2^{|S_1 \backslash \sigma_k|}$. By the parallelogram law and the fact that $|S_1\backslash\sigma_k| \le |S_1|/2^k$, we have

$$\operatorname*{Ave}_{(\delta_j)\in J_k}\left|\sum_{j\in S_1}\delta_j u_j\right|_n^2 = 2^k\sum_{j\in S_1\backslash\sigma_k}|u_j|_n^2 \le |S_1|,$$

and Markov's inequality implies that there exists $M^{k+1}\subseteq J_k\cap\sqrt{2|S_1|}\mathcal{E}_{S_1}$ with $|M^{k+1}|\ge 2^{|S_1\backslash\sigma_k|-1}$. Then, by Lemma 1, we can find $\sigma^*_{k+1}\subseteq S_1\backslash\sigma_k$, of cardinality $|\sigma^*_{k+1}|\ge\frac{|S_1\backslash\sigma_k|}{2}$, for which

$$P_{\sigma_k\cup\sigma^*_{k+1}}(M^{k+1}) = \mathbf{0}_{\sigma_k}\times\{-2^{k/2},2^{k/2}\}^{\sigma^*_{k+1}}.$$

Since $M^{k+1}\subseteq\sqrt{2|S_1|}\mathcal{E}_{S_1}\cap 2^{k/2}Q_{S_1}$, it follows that

$$(**)\qquad \mathbf{0}_{\sigma_k}\times 2^kQ_{\sigma^*_{k+1}}\subseteq P_{\sigma_k\cup\sigma^*_{k+1}}(2^{k/2}\sqrt{2|S_1|}\mathcal{E}_{S_1}\cap 2^kQ_{S_1}).$$

Suppose now that $a\in Q_{\sigma_k}$, $b\in Q_{\sigma^*_{k+1}}$. From the inductive hypothesis $(*)$, there exists $t_a\in\alpha_k\sqrt{2|S_1|}\mathcal{E}_{S_1}\cap\beta_kQ_{S_1}$ such that $P_{\sigma_k}(t_a)=a$. Let $w_a=P_{\sigma^*_{k+1}}(t_a)$; then $w_a\in\beta_kQ_{\sigma^*_{k+1}}$ and

$$(a,w_a)\in P_{\sigma_k\cup\sigma^*_{k+1}}(\alpha_k\sqrt{2|S_1|}\mathcal{E}_{S_1}\cap\beta_kQ_{S_1}).$$

If $v_{a,b}=b-w_a$, it is clear that $v_{a,b}\in Q_{\sigma^*_{k+1}}+\beta_kQ_{\sigma^*_{k+1}}=2^kQ_{\sigma^*_{k+1}}$, and therefore, by $(**)$,

$$(\mathbf{0}_{\sigma_k},v_{a,b})\in P_{\sigma_k\cup\sigma^*_{k+1}}(2^{k/2}\sqrt{2|S_1|}\mathcal{E}_{S_1}\cap 2^kQ_{S_1}).$$

Then,

$$\begin{aligned}(a,b)=&(a,w_a)+(\mathbf{0}_{\sigma_k},v_{a,b})\in P_{\sigma_k\cup\sigma^*_{k+1}}(\alpha_k\sqrt{2|S_1|}\mathcal{E}_{S_1}\cap\beta_kQ_{S_1})\\ &+P_{\sigma_k\cup\sigma^*_{k+1}}(2^{k/2}\sqrt{2|S_1|}\mathcal{E}_{S_1}\cap 2^kQ_{S_1})\\ \subseteq&P_{\sigma_k\cup\sigma^*_{k+1}}(\alpha_{k+1}\sqrt{2|S_1|}\mathcal{E}_{S_1}\cap\beta_{k+1}Q_{S_1}).\end{aligned}$$

Since $a\in Q_{\sigma_k}, b\in Q_{\sigma^*_{k+1}}$ were arbitrary, this means that

$$Q_{\sigma_k\cup\sigma^*_{k+1}}\subseteq P_{\sigma_k\cup\sigma^*_{k+1}}(\alpha_{k+1}\sqrt{2|S_1|}\mathcal{E}_{S_1}\cap\beta_{k+1}Q_{S_1}).$$

If we define $\sigma_{k+1}=\sigma_k\cup\sigma^*_{k+1}$, we readily see that $|\sigma_{k+1}|\ge(1-\frac{1}{2^{k+1}})|S_1|$, and this completes the inductive step. The first step $(k=1)$ is much simpler. ☐

For our next two lemmas we shall need to assume that the vectors $u_1,\dots,u_s$ are linearly independent.

Lemma 3. *Let S_1 be a non-empty subset of S. Then, for every $\theta\in(0,\frac14)$, we can find disjoint $\sigma,\tau\subseteq S_1$ with $|\sigma|\ge\frac{|S_1|}{2}, |\tau|\le\theta|S_1|$, and*

$$P_{S_1\backslash\tau}(\mathcal{E}_{S_1})\supseteq\mathbf{0}_{S_1\backslash(\sigma\cup\tau)}\times c\sqrt{\theta}D_\sigma,$$

where $c>0$ is an absolute constant.

Proof. Set $V_{S_1} = \operatorname{span}\{u_j, j \in S_1\}$. Then, there exist $x_i \in V_{S_1}, i \in S_1$, such that

$$\langle x_i, u_j \rangle = \delta_{ij} \quad \text{for any pair of } i, j \in S_1.$$

Applying Lemma 2 for the ellipsoid $\mathcal{E}_{S_1}$, we obtain $\tau \subseteq S_1, |\tau| \leq \theta |S_1|$, for which

$$P_{S_1 \setminus \tau}(\mathcal{E}_{S_1}) \supseteq \frac{c\sqrt{\theta}}{\sqrt{|S_1|}}[-1,1]^{S_1 \setminus \tau}.$$

Then, for any choice of scalars $(t_i)_{i \in S_1 \setminus \tau}$, we can find a vector $(\delta_j)_{j \in S_1}$ in $\mathcal{E}_{S_1}$ whose restriction in $\mathbf{R}^{S_1 \setminus \tau}$ is $(\frac{c\sqrt{\theta}}{\sqrt{|S_1|}} \operatorname{sign} t_j)_{j \in S_1 \setminus \tau}$. In view of the orthogonality relations between the x_i's and the u_j's we see that

$$\begin{aligned}
\sum_{i \in S_1 \setminus \tau} |t_i| &= \left\langle \sum_{i \in S_1 \setminus \tau} t_i x_i, \sum_{j \in S_1 \setminus \tau} (\operatorname{sign} t_j) u_j \right\rangle \\
&= \frac{\sqrt{|S_1|}}{c\sqrt{\theta}} \left\langle \sum_{i \in S_1 \setminus \tau} t_i x_i, \sum_{j \in S_1} \delta_j u_j \right\rangle \\
&\leq \frac{\sqrt{|S_1|}}{c\sqrt{\theta}} \left| \sum_{i \in S_1 \setminus \tau} t_i x_i \right|_n \left| \sum_{j \in S_1} \delta_j u_j \right|_n \\
&\leq \frac{\sqrt{|S_1|}}{c\sqrt{\theta}} \left| \sum_{i \in S_1 \setminus \tau} t_i x_i \right|_n .
\end{aligned}$$

It follows that the operator $T\colon \operatorname{span}\{x_i, i \in S_1 \setminus \tau\} \subset l_2^n \to l_1^{|S_1 \setminus \tau|}$, defined by $Tx_i = e_i$ (where $\{e_i\}$ is the canonical orthonormal basis in $\mathbf{R}^{|S_1 \setminus \tau|}$), has norm not exceeding $\sqrt{|S_1|}/c\sqrt{\theta}$. Then, $T^*\colon l_\infty^{|S_1 \setminus \tau|} \to l_2^n$ is a 2-absolutely summing operator with 2-summing norm $\pi_2(T^*) \leq K_G \frac{\sqrt{|S_1|}}{c\sqrt{\theta}}$, where K_G is Grothendieck's constant. From Pietch's factorization theorem, applied in the same context as in the proof of Theorem 1.2 [B-T], we can find positive real numbers $\lambda_i, i \in S_1 \setminus \tau$, with $\sum_{i \in S_1 \setminus \tau} \lambda_i^2 = 1$, such that, for any reals $t_i, i \in S_1 \setminus \tau$,

$$\left(\sum_{i \in S_1 \setminus \tau} \left(\frac{t_i}{\lambda_i} \right)^2 \right)^{1/2} \leq K_G \frac{\sqrt{|S_1|}}{c\sqrt{\theta}} \left| \sum_{i \in S_1 \setminus \tau} t_i x_i \right|_n .$$

Since $\sum_{i \in S_1 \setminus \tau} \lambda_i^2 = 1$ and $\theta < \frac{1}{4}$, we apply Markov's inequality to obtain $\sigma \subseteq S_1 \setminus \tau, |\sigma| \geq \frac{|S_1|}{2}$, with $\lambda_i \leq \frac{2}{\sqrt{|S_1|}}$ for every $i \in \sigma$. Suppose now that $(\delta_j)_{j \in \sigma} \in D_\sigma$ is given, i.e. $\sum_{j \in \sigma} \delta_j^2 \leq 1$. The set $\{u_j, j \in \tau\} \cup \{x_i, i \in S_1 \setminus \tau\}$ is linearly independent (hence a basis) in V_{S_1}, so we can write

$$\sum_{j \in \sigma} \delta_j u_j + \sum_{j \in \tau} \rho_j u_j = \sum_{i \in S_1 \setminus \tau} t_i x_i$$

for suitable $(\rho_j)_{j\in\tau}$, $(t_i)_{i\in S_1\setminus\tau}$. Then,

$$\begin{aligned} \left|\sum_{j\in\sigma}\delta_j u_j+\sum_{j\in\tau}\rho_j u_j\right|_n^2 &= \left\langle \sum_{j\in\sigma}\delta_j u_j+\sum_{j\in\tau}\rho_j u_j, \sum_{i\in S_1\setminus\tau} t_i x_i\right\rangle \\ &= \left\langle \sum_{j\in\sigma}\delta_j u_j, \sum_{i\in\sigma} t_i x_i\right\rangle = \sum_{i\in\sigma}\delta_i t_i \le \left(\sum_{i\in\sigma} t_i^2\right)^{1/2} \\ &\le \left(\sum_{i\in\sigma}\left(\frac{t_i}{\lambda_i}\right)^2\right)^{1/2}\frac{2}{\sqrt{|S_1|}} \le \left(\sum_{i\in S_1\setminus\tau}\left(\frac{t_i}{\lambda_i}\right)^2\right)^{1/2}\frac{2}{\sqrt{|S_1|}} \\ &\le \frac{2K_G}{c}\frac{1}{\sqrt{\theta}}\left|\sum_{i\in S_1\setminus\tau} t_i x_i\right|_n , \end{aligned}$$

and therefore,

$$\left|\sum_{j\in\sigma}\delta_j u_j+\sum_{j\in\tau}\rho_j u_j\right|_n \le \frac{2K_G}{c}\frac{1}{\sqrt{\theta}}.$$

This means that $\mathbf{0}_{S_1\setminus(\sigma\cup\tau)}\times(\delta_j)_{j\in\sigma}\in\frac{1}{c_1\sqrt{\theta}}P_{S_1\setminus\tau}(\mathcal{E}_{S_1})$ with $c_1=c/2K_G$. Since $(\delta_j)_{j\in\sigma}$ was arbitrary in D_σ, the lemma follows. □

We are now ready to prove Theorem 2 in the case of independent u_j's:

Lemma 4. *For every $\varepsilon\in(0,1)$ one can find $\sigma\subseteq S, |\sigma|\geq(1-\varepsilon)s$, such that*

$$P_\sigma(\mathcal{E})\supseteq c\sqrt{\varepsilon}D_\sigma,$$

where $c>0$ is an absolute constant.

Proof. Given $\varepsilon\in(0,1)$, we set $\theta=\varepsilon/7$. Let also k be the non-negative integer for which $\frac{1}{2^{k+1}}\leq\varepsilon<\frac{1}{2^k}$. To obtain σ, we shall follow an inductive procedure based on Lemma 3:

Step 1: We set $S_0=S$, and $\theta_1=\theta$. Since $\theta_1\in(0,\frac{1}{4})$, we can find a pair (σ_1,τ_1) of disjoint subsets of S_0, with $|\tau_1|\leq\theta_1|S_0|, |\sigma_1|\geq\frac{1}{2}|S_0|$, and $P_{S_0\setminus\tau_1}(\mathcal{E}_{S_0})\supseteq\mathbf{0}_{S_0\setminus(\sigma_1\cup\tau_1)}\times c\sqrt{\theta_1}D_{\sigma_1}$, where c is the constant from Lemma 3. Finally, we define $S_1=S_0\setminus(\sigma_1\cup\tau_1)$. Note that $|S_1|\leq\frac{1}{2}|S_0|=\frac{s}{2}$.

Inductive step: Suppose that S_l has been defined, and $|S_l|>\frac{\varepsilon}{2}s$. If, in addition, $l<k+2$, we define $\theta_{l+1}=2^{l/2}\theta$. Note that then $\theta_{l+1}\leq\frac{\sqrt{2}}{7}2^{k/2}\varepsilon<\frac{\sqrt{2}}{7}\sqrt{\varepsilon}<\frac{1}{4}$, and therefore we can apply Lemma 3 for $\mathcal{E}_{S_l}$ and θ_{l+1} to obtain a pair $(\sigma_{l+1},\tau_{l+1})$ of disjoint subsets of S_l, with $|\tau_{l+1}|\leq\theta_{l+1}|S_l|, |\sigma_{l+1}|\geq\frac{1}{2}|S_l|$, and $P_{S_l\setminus\tau_{l+1}}(\mathcal{E}_{S_l})\supseteq\mathbf{0}_{S_l\setminus(\sigma_{l+1}\cup\tau_{l+1})}\times c\sqrt{\theta_{l+1}}D_{\sigma_{l+1}}$. To complete the inductive step, we define $S_{l+1}=S_l\setminus(\sigma_{l+1}\cup\tau_{l+1})$. Note also that $|S_{l+1}|\leq\frac{1}{2}|S_l|$, hence, as far as we continue performing these steps, $|S_l|\leq\frac{s}{2^l}$.

We end this inductive construction when we arrive at a set S_l of cardinality $|S_l|\leq\frac{\varepsilon}{2}s$. This will certainly happen after at most $(k+2)$-steps, since $\frac{1}{2^{k+2}}\leq\frac{\varepsilon}{2}$ and our construction implies that $|S_l|<\frac{s}{2^l}$ for every admissible l.

Suppose l_* is the first index for which $|S_{l_*}|\leq\frac{\varepsilon}{2}s$. We define $\sigma=\sigma_1\cup\cdots\cup\sigma_{l_*}$.

Claim 1. $|\sigma| \geq (1-\varepsilon)s$.

[*Proof.* Note that $\bigcup_{1\leq l\leq l_*}(\sigma_l \cup \tau_l) = S\backslash S_{l_*}$, hence

$$|S\backslash\sigma| = |S_{l_*}| + \sum_{1\leq l\leq l_*} |\tau_l| \leq \frac{\varepsilon}{2}s + \sum_{1\leq l\leq l_*} \theta_l |S_{l-1}|$$

$$\leq \frac{\varepsilon}{2}s + \theta \sum_{1\leq l\leq l_*} 2^{(l-1)/2}\frac{s}{2^{l-1}} < \frac{\varepsilon}{2}s + \frac{\varepsilon}{7}s\left(\sum_{l=0}^{\infty}\frac{1}{2^{l/2}}\right) < \varepsilon s.]$$

Claim 2. If $1 \leq l \leq l_*$, then

$$P_\sigma(\mathcal{E}) \supseteq \mathbf{0}_{\sigma\backslash\sigma_l} \times c2^{(l-1)/4}\sqrt{\theta}D_{\sigma_l}.$$

[*Proof.* Suppose that $\Delta_l = (\delta_j)_{j\in\sigma_l} \in c2^{(l-1)/4}\sqrt{\theta}D_{\sigma_l}$. Then our construction implies that $\mathbf{0}_{S_{l-1}\backslash(\sigma_l\cup\tau_l)} \times \Delta_l \in P_{S_{l-1}\backslash\tau_l}(\mathcal{E}_{S_{l-1}})$. Hence, we can find $(\zeta_i)_{i\in\tau_l}$ such that

$$\left|\sum_{j\in\sigma_l}\delta_j u_j + \sum_{i\in\tau_l}\zeta_i u_i\right|_n \leq 1.$$

Since $\sigma \cap \tau_l = \emptyset$, it is clear that $\mathbf{0}_{\sigma\backslash\sigma_l} \times \Delta_l \in P_\sigma(\mathcal{E})$.]

To conclude the proof of the lemma, suppose that $\Delta = (\delta_j)_{j\in\sigma}$ is an arbitrary point in D_σ, i.e. $\sum_{j\in\sigma}\delta_j^2 \leq 1$. Consider the restriction $\Delta_l = \mathbf{0}_{\sigma\backslash\sigma_l} \times (\delta_j)_{j\in\sigma_l}$ of Δ in $\mathbf{R}^{\sigma_l}$, and set $|\Delta_l| = (\sum_{j\in\sigma_l}\delta_j^2)^{1/2}$, $1 \leq l < l_*$. By Claim 2, each Δ_l belongs to $\frac{|\Delta_l|}{c2^{(l-1)/4}\sqrt{\theta}}P_\sigma(\mathcal{E})$; thus

$$\Delta = \sum_{1\leq l\leq l_*}\Delta_l \in \left(\sum_{1\leq l\leq l_*}\frac{|\Delta_l|}{c2^{(l-1)/4}\sqrt{\theta}}\right)P_\sigma(\mathcal{E})$$

$$\subseteq \frac{1}{c\sqrt{\theta}}\left(\sum_{1\leq l\leq l_*}|\Delta_l|^2\right)^{1/2}\left(\sum_{l=0}^{\infty}\frac{1}{2^{l/2}}\right)^{1/2}P_\sigma(\mathcal{E})$$

$$\subseteq \frac{\sqrt{7}}{c}\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)^{1/2}\frac{1}{\sqrt{\varepsilon}}P_\sigma(\mathcal{E}),$$

and the lemma is proved with $c' = c/5$. □

Proof of Theorem 2. Assume that $u_1, \ldots, u_s$ are arbitrary vectors in $\mathbf{R}^n$ with $|u_j|_n \leq 1, j = 1, \ldots, s$. Set $v_j = u_j + e_{j+n}, j = 1, \ldots, s$, where $\{e_i\}_{i\leq n+s}$ is the canonical orthonormal basis in $\mathbf{R}^{n+s}$. Then, the v_j's are linearly independent vectors in $\mathbf{R}^{n+s}$, of Euclidean norm at most $\sqrt{2}$, and if

$$\mathcal{E}^* = \left\{(\delta_j)_{j\leq s} \colon \left|\sum_{j\leq s}\delta_j v_j\right|_{n+s} \leq 1\right\},$$

Lemma 4 implies that, given $\varepsilon \in (0,1)$, there exists $\sigma \subseteq S, |\sigma| \geq (1-\varepsilon)s$, for which

$$P_\sigma(\mathcal{E}^*) \supseteq c''\sqrt{\varepsilon}D_\sigma$$

with $c'' = c/\sqrt{2}, c$ the constant from Lemma 4. Since

$$|\sum_{j\leq s}\delta_j v_j|_{n+s}^2 = |\sum_{j\leq s}\delta_j u_j|_n^2 + \sum_{j\leq s}\delta_j^2,$$

we readily see that

$$P_\sigma(\mathcal{E}) \supseteq c''\sqrt{\varepsilon} D_\sigma$$

and Theorem 2 is proved. □

3. Proof of Theorem 1 with $d = 1$

For the proof of the proportional Dvoretzky-Rogers factorization result, we shall combine Theorem 2 with the method used in [S-T]: Let $X = (\mathbf{R}^n, \|\cdot\|)$ be an n-dimensional normed space and $\varepsilon \in (0,1)$ be given. Without loss of generality, we may assume that the ellipsoid of minimal volume containing the unit ball B_X of X is the Euclidean unit ball D. By John's theorem [J], $D \subseteq \sqrt{n} B_X$. We can also find contact points y_i, $i \le N$, $\|y_i\|_X = |y_i|_n = 1$, $N = O(n^2)$, and positive real numbers μ_i, $i \le N$, such that the following representation of the identity holds: for every $x \in \mathbf{R}^n$, $x = \sum_{i \le N} \mu_i \langle x, y_i \rangle y_i$. Now, if s is the smallest integer $\ge (1 - \frac{\varepsilon}{2})n$, we can choose $x_1, \ldots, x_s$ among the y_i's so that:

Lemma 5 ([S-T])**.** $\mathrm{dist}(x_i, \mathrm{span}\{x_j, j \ne i\}) \ge \sqrt{\frac{\varepsilon}{2}}$, $i = 1, \ldots, s$.

Hence, there exist v_j, $j \le s$, in $\mathrm{span}\{x_i, i \le s\}$ satisfying
(i) $|v_j|_n \le \sqrt{2/\varepsilon}$, $j = 1, \ldots, s$,
(ii) $\langle x_i, v_j \rangle = \delta_{ij}, i, j = 1, \ldots, s$.

Set $u_j = \sqrt{\varepsilon/2} v_j$ and define $\mathcal{E} = \{(\delta_j)_{j \le s} \colon |\sum_{j \le s} \delta_j u_j|_n \le 1\}$. From Theorem 2 we obtain $\sigma \subseteq S, |\sigma| \ge (1 - \frac{\varepsilon}{2})s$, with $P_\sigma(\mathcal{E}) \supseteq c\sqrt{\varepsilon} D_\sigma$. Then $|\sigma| \ge (1 - \varepsilon)n$, and for any choice of scalars $\mathbf{t} = (t_i)_{i \in \sigma}$ we have

$$|\mathbf{t}|^2 = \sum_{i \in \sigma} t_i^2 = \left\langle \sum_{i \in \sigma} t_i x_i, \sum_{j \in \sigma} t_j v_j \right\rangle = \sqrt{\frac{2}{\varepsilon}} \left\langle \sum_{i \in \sigma} t_i x_i, \sum_{j \in \sigma} t_j u_j \right\rangle .$$

We can extend $(\frac{c\sqrt{\varepsilon}}{|\mathbf{t}|} t_j)_{j \in \sigma}$ to a vector $(\delta_j)_{j \le s}$ in $\mathcal{E}$. Hence,

$$|\mathbf{t}|^2 = \sqrt{\frac{2}{\varepsilon}} \frac{|\mathbf{t}|}{c\sqrt{\varepsilon}} \left\langle \sum_{i \in \sigma} t_i x_i, \sum_{j \le s} \delta_j u_j \right\rangle \le \frac{c'}{\varepsilon} |\mathbf{t}| \left| \sum_{i \in \sigma} t_i x_i \right|_n$$

and, since $|\cdot|_n \le \|\cdot\|_X$ and the x_i's are of $\|\cdot\|$-norm one, we have

$$\left(\sum_{i \in \sigma} t_i^2 \right)^{1/2} \le \frac{c'}{\varepsilon} \left| \sum_{i \in \sigma} t_i x_i \right|_n \le \frac{c'}{\varepsilon} \left\| \sum_{i \in \sigma} t_i x_i \right\|_X \le \frac{c'}{\varepsilon} \sum_{i \in \sigma} |t_i|.$$

Defining $\beta \colon l_1^{|\sigma|} \to X$ with $\beta(e_i) = x_i$, $i \in \sigma$, and $\alpha \colon X \to l_2^{|\sigma|}$ with $\alpha = TP_\sigma$ where P_σ is the orthogonal projection of X onto $\mathrm{span}\{x_i, i \in \sigma\}$ and $Tx_i = e_i$, we have a factorization $i_{1,2} = \alpha \circ \beta$ of the identity $i_{1,2} \colon l_1^{|\sigma|} \to l_2^{|\sigma|}$ with $\|\alpha\| \cdot \|\beta\| \le c'/\varepsilon$. By duality and by the extension property of l_∞^n, this is then equivalent to the assertion of the theorem.

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References

[B-S] J. Bourgain and S. J. Szarek, *The Banach-Mazur distance to the cube and the Dvoretzky-Rogers factorization*, Israel J. Math. **62** 1988, 169–180. MR **89g:**46026

[B-T] J. Bourgain and L. Tzafriri, *Invertibility of "large" submatrices with applications to the geometry of Banach spaces and harmonic analysis*, Israel J. Math. **57** (1987), 137–224. MR **89a:**46035

[D-R] A. Dvoretzky and C. A. Rogers, *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 192–197. MR **11:**525a

[G] A. A. Giannopoulos, *A note on the Banach-Mazur distance to the cube*, GAFA Seminar (to appear).

[J] F. John, *Extremum problems with inequalities as subsidiary conditions*, Courant Anniversary Volume, Interscience, New York, 1948. MR **10:**719b

[L-T] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces* I: *Sequence spaces*, Springer-Verlag, Berlin and New York, 1977. MR **58:**17766

[M-Sc] V. D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math., vol. 1200, Springer-Verlag, Berlin and New York, 1986. MR **87m:**46038

[Pi] A. Pietsch, *Operator ideals*, North-Holland, Amsterdam, 1978. MR **81a:**47002

[S-T] S. J. Szarek and M. Talagrand, *An "isomorphic" version of the Sauer-Shelah lemma and the Banach-Mazur distance to the cube*, GAFA Seminar 87-88, Lecture Notes in Math., vol. 1376, Springer-Verlag, Berlin and New York, 1989, 105–112. MR **90h:**46034

[Sa] N. Sauer, *On the density of families of sets*, J. Combin. Theory Ser. A **13** (1972), 145–147. MR **46:**7017

[Sh] S. Shelah, *A combinatorial problem: stability and order for models and theories in infinitary languages*, Pacific J. Math. **41** (1972), 247–261. MR **46:**7018

[Sz.1] S. J. Szarek, *Spaces with large distance to* l_∞^n *and random matrices*, Amer. J. Math. **112** (1990), 899–942. MR **91j:**46023

[Sz.2] ———, *On the geometry of the Banach-Mazur compactum*, Lecture Notes in Math., vol. 1470, Springer-Verlag, Berlin and New York, 1991, pp. 48–59. MR **93b:**46019

[T-J] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite dimensional operator ideals*, Longman Sci. Tech., Harlow, 1988. MR **90k:**46039

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