

CLASS NUMBERS AND IWASAWA INVARIANTS OF QUADRATIC FIELDS

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ABSTRACT. Let $\mathbf{Q}(\sqrt{-d})$ and $\mathbf{Q}(\sqrt{3d})$ be quadratic fields with $d \equiv 2 \pmod{3}$ a positive integer. Let λ^-, λ^+ be the respective Iwasawa λ -invariants of the cyclotomic \mathbf{Z}_3 -extension of these fields. We show that if $\lambda^- = 1$, then 3 does not divide the class number of $\mathbf{Q}(\sqrt{3d})$ and $\lambda^+ = 0$.

INTRODUCTION

Let $k^- = \mathbf{Q}(\sqrt{-d})$ and $k^+ = \mathbf{Q}(\sqrt{3d})$ with d a positive integer. In [5], Washington showed that constraints on the 3-Sylow subgroup and the fundamental unit of k^+ force λ^- to be 1, where λ^- is the Iwasawa λ -invariant associated to the cyclotomic \mathbf{Z}_3 -extension of k^- . Here, using similar methods, we show that if $\lambda^- = 1$ and 3 splits in k^- , then 3 does not divide the class number of k^+ . Since recent results of Jochnowitz [3, 4] imply that there are infinitely many imaginary quadratic fields in which 3 splits and $\lambda^- = 1$, we obtain as a corollary that there are infinitely many real quadratic fields $\mathbf{Q}(\sqrt{3d})$ with 3 splitting in $\mathbf{Q}(\sqrt{-d})$ such that the class number of $\mathbf{Q}(\sqrt{3d})$ is relatively prime to 3. (We note that Horie [1] has proven a similar result concerning real quadratic fields by requiring that 3 neither divides the class number of nor splits in $\mathbf{Q}(\sqrt{-d})$.)

It then follows that there are infinitely many real quadratic fields with $\lambda^+ = 0$, where λ^+ is the Iwasawa λ -invariant associated to the \mathbf{Z}_3 -extension of k^+ .

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We begin with a brief review of p -adic L -functions. For more details, see [6]. Let p be an odd prime and let \mathbf{Z}_p , \mathbf{Q}_p and \mathbf{C}_p denote the p -adic integers, the p -adic rationals and the completion of the algebraic closure of \mathbf{Q}_p respectively. Let ω denote the Teichmüller character and let ψ be a primitive Dirichlet character of conductor f , with p^2 not dividing f . We let $d = f$ if p does not divide f and $d = \frac{f}{p}$ if p does divide f . The generalized Bernoulli number $B_{n,\psi}$ is defined by

$$\sum_{a=1}^f \frac{\psi(a)e^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\psi} \frac{t^n}{n!}.$$

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The p -adic L -function $L_p(s, \psi)$ is the unique meromorphic p -adic function $\mathbf{Z}_p \rightarrow \mathbf{C}_p$ which for $n \geq 1$ satisfies

$$(1) \quad L_p(1-n, \psi) = -(1 - \psi\omega^{-n}(p)p^{n-1}) \frac{B_{n, \psi\omega^{-n}}}{n}.$$

In order to ensure that $L_p(s, \psi)$ is not identically zero, we now assume that ψ is a non-trivial even character. If $O_\psi = \mathbf{Z}_p[\psi(1), \psi(2), \dots]$, Iwasawa has shown that there is a power series $F(T, \psi) \in O_\psi[[T]]$ such that

$$L_p(s, \psi) = F((1+pd)^s - 1, \psi).$$

From the p -adic Weierstrass Preparation Theorem [6] we see that $F(T, \psi) = G(T)U(T)$ where $U(T)$ is a unit of $O_\psi[[T]]$ and $G(T)$ is a distinguished polynomial. Then, $G(T) = a_0 + a_1T + \dots + a_{n-1}T^{\lambda-1} + T^\lambda$ and if π generates the ideal of O_ψ lying over p , then π divides a_i , $0 \leq i \leq \lambda - 1$. We note that if ψ is an even quadratic character and $p = 3$, then λ is related to the class group of certain number fields. That is, we let k be the imaginary quadratic field associated to $\psi\omega^{-1}$ with k_∞ its cyclotomic \mathbf{Z}_3 -extension. Also, let k_n be the unique subfield of k_∞ of degree 3^n over k and let A_n be the 3-Sylow subgroup of k_n . Then, via the natural injection $A_n \rightarrow A_{n+1}$ for all $n \geq 0$,

$$\bigcup_{n \geq 0} A_n \cong (\mathbf{Q}_3/\mathbf{Z}_3)^\lambda.$$

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Let K be a real quadratic field with character χ , fundamental unit ϵ , discriminant D and class number h^+ . Leopoldt's p -adic class number formula says that

$$(2) \quad \frac{2h^+ \log_p(\epsilon)}{\sqrt{D}} \left(1 - \frac{\chi(p)}{p}\right) = L_p(1, \chi)$$

where \log_p denotes the p -adic logarithm.

We now assume that $p = 3$ and let λ^- (resp. λ^+) be the Iwasawa λ -invariant associated to the cyclotomic \mathbf{Z}_3 -extension of $\mathbf{Q}(\sqrt{-d})$ (resp. $\mathbf{Q}(\sqrt{3d})$) for the prime 3.

Theorem. *Assume $d \equiv 2 \pmod{3}$ and $\lambda^- = 1$. Then 3 does not divide the class number of $\mathbf{Q}(\sqrt{3d})$. In particular, $\lambda^+ = 0$.*

Proof. Let χ be the non-trivial even quadratic character of conductor $3d$. Since 3 splits in $\mathbf{Q}(\sqrt{-d})$, we have that $L_3(0, \chi) = 0$ from (1). Furthermore, since $\lambda^- = 1$, $F(T, \chi) = (b_0 + b_1T)U(T)$ where $U(T)$ is a unit of $\mathbf{Z}_3[[T]]$ and b_1 is a 3-adic unit. Because $L_3(0, \chi) = F(0, \chi)$, we have $F(T, \chi) = (b_1T)U(T)$. Since 3 does not divide b_1 , $L_3(1, \chi) \not\equiv 0 \pmod{9}$. (See Lemma 1 of [5].) Then,

$$\frac{2h^+ \log_3(\epsilon)}{\sqrt{D}} \not\equiv 0 \pmod{9}$$

from (2). Thus, in order to prove that 3 does not divide h^+ , it suffices to show that $\log_3(\epsilon) \equiv 0 \pmod{3\sqrt{3d}}$. In order to prove this congruence, note that the

3-integrality of $L_3(1, \chi)$ together with the fact that in this situation there is a $\sqrt{3d}$ in the denominator of (2) imply that $\log_3(\epsilon)$ must have half-integral (non-integral) 3-adic valuation. Thus, it is sufficient to show that $\log_3(\epsilon) \equiv 0 \pmod{3}$. Let

$$\epsilon = a + b\sqrt{3d} \text{ or } \frac{a + b\sqrt{3d}}{2}.$$

Then

$$\epsilon^2 - 1 \equiv 2ab\sqrt{3d} \pmod{3}.$$

Since

$$\log_3(\epsilon^2) = \log_3(\epsilon^2 - 1 + 1) \equiv (\epsilon^2 - 1) - \frac{(\epsilon^2 - 1)^2}{2} + \frac{(\epsilon^2 - 1)^3}{3} \pmod{3},$$

we see that

$$\log_3(\epsilon^2) \equiv 2ab\sqrt{3d} + 8a^3b^3d\sqrt{3d} \pmod{3}.$$

Since $d \equiv 2 \pmod{3}$, we see that $\log_3(\epsilon^2) \equiv 0 \pmod{3}$. Thus, $\log_3(\epsilon) \equiv 0 \pmod{3}$ as well.

Finally, a theorem of Iwasawa [2] says that if 3 totally ramifies in the cyclotomic \mathbf{Z}_3 -extension of $\mathbf{Q}(\sqrt{3d})$ and h^+ is not divisible by 3, then $\lambda^+ = 0$.

In [4], Jochowitz proves that given an arbitrary odd prime p , there are infinitely many imaginary quadratic fields in which p splits and whose Iwasawa λ -invariant associated to p equals 1. This immediately implies the following.

Corollary. *There are infinitely many real quadratic fields which have class number not divisible by 3 and whose Iwasawa λ -invariant associated to 3 equals zero.*

Examples. We now give several examples. The first illustrates our theorem, the next two show that if $d \not\equiv 2 \pmod{3}$ and $\lambda^- = 1$, then it is possible to have 3 dividing h^+ , and the final one shows that if $d \equiv 2 \pmod{3}$ and $\lambda^- \neq 1$, then it is also possible to have 3 dividing h^+ .

d	λ^-	h^+
23	1	1
237	1	3
262	1	6
107	2	3

REFERENCES

1. K. Horie, *A note on basic Iwasawa λ -invariants of imaginary quadratic fields*, Invent. Math. **88** (1987), 31–38. MR **88i**:11073
2. K. Iwasawa, *A note on class numbers of algebraic number fields*, Abh. Math. Sem. Univ. Hamburg **20** (1956), 257–258. MR **18**:644d
3. N. Jochowitz, *A p -adic conjecture about derivatives of L -series attached to modular forms*, Proceedings of the Boston University Conference on p -Adic Monodromy and the p -Adic Birch and Swinnerton-Dyer Conjecture (to appear). CMP 94:13

4. ———, *An alternative approach to non-vanishing theorems for coefficients of half integral weight forms mod p and implications for Iwasawa's λ -invariant for quadratic fields* (to appear).
5. L. Washington, *Zeros of p -adic L -functions*, Sém Delange-Pisot-Poitou, Théorie des Nombres, 1980/1981, Birkhäuser, Boston, Basel, and Stuttgart, 1982. MR **84f**:12008
6. ———, *Introduction to cyclotomic fields*, Graduate Texts in Math., Springer-Verlag, New York, 1982. MR **85g**:11001

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