

A CARTAN THEOREM FOR BANACH ALGEBRAS

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ABSTRACT. Let A be a semisimple Banach algebra, and let Ω_A be its spectral unit ball. We show that every holomorphic map $G: \Omega_A \rightarrow \Omega_A$ satisfying $G(0) = 0$ and $G'(0) = I$ fixes those elements of Ω_A which belong to the centre of A , but not necessarily any others. Using this, we deduce that the automorphisms of Ω_A all leave the centre invariant. As a further application, we give a new proof of Nagasawa's generalization of the Banach-Stone theorem.

1. STATEMENT OF RESULTS

Given a (complex, unital) Banach algebra A , its *spectral unit ball* is the set

$$\Omega_A := \{a \in A: \rho(a) < 1\},$$

where $\rho(a)$ denotes the spectral radius of a . It is always an open subset of A , though, unlike its norm counterpart, it need be neither convex nor bounded. It arises, for example, in the spectral Nevanlinna-Pick problem (see e.g. [3]), which is of importance in control theory.

A major problem is to identify the holomorphic automorphisms of Ω_A , these being the natural analogue of the Möbius automorphisms of the unit disc. In [5], M. C. White and the author studied these automorphisms in the important special case when $A = \mathcal{M}_n$, the algebra of $n \times n$ matrices. The main result was that, in contrast with the situation for the unit disc, the automorphism group is *not* transitive on $\Omega_{\mathcal{M}_n}$ if $n \geq 2$. The proof of this was rather intricate and depended on the special properties of matrices. In this paper we shall give a much simpler proof, which at the same time extends to more general Banach algebras. We shall also see that there is a link with a long-standing open problem in Banach-algebra theory.

Our principal tool is the following analogue of Cartan's theorem. We write I for the identity map, and Z_A for the centre of a Banach algebra A :

$$Z_A := \{c \in A: ac = ca \text{ for all } a \in A\}.$$

Theorem 1. *Let A be a semisimple Banach algebra.*

- (a) *If $G: \Omega_A \rightarrow \Omega_A$ is a holomorphic map satisfying $G(0) = 0$ and $G'(0) = I$, then $G(c) = c$ for all $c \in \Omega_A \cap Z_A$.*

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- (b) Given $a \in \Omega_A \setminus Z_A$, there exists a holomorphic map $G: \Omega_A \rightarrow \Omega_A$ satisfying $G(0) = 0$ and $G'(0) = I$ such that $G(a) \neq a$.

The reason why the classical Cartan theorem does not apply to give $G \equiv I$ on the whole of Ω_A is that, as remarked earlier, Ω_A may be unbounded. In fact, using [1, Corollary 5.2.3] one can easily show that Ω_A is bounded if and only if A is isomorphic to a uniform algebra.

Using Theorem 1, we deduce our result on non-transitivity.

Theorem 2. *Let A and B be semisimple Banach algebras. If $F: \Omega_A \rightarrow \Omega_B$ is a biholomorphic map, then $F(\Omega_A \cap Z_A) = \Omega_B \cap Z_B$.*

Thus, in particular, if $A = \mathcal{M}_n$ and F is an automorphism of $\Omega_{\mathcal{M}_n}$, then $F(0)$ is necessarily a multiple of the identity, thereby re-proving the main result of [5].

Returning to Theorem 1, even though we cannot conclude that G fixes points of $\Omega_A \setminus Z_A$, it is still possible to say something about its action there. This is a consequence of the following version of Schwarz's lemma. We write $\sigma(a)$ for the spectrum of a , and Γ_r for the circle in \mathbb{C} with centre 0 and radius r .

Theorem 3. *Let A and B be Banach algebras, and let $F: \Omega_A \rightarrow \Omega_B$ be a holomorphic map such that $F(0) = 0$. Then for each $a \in \Omega_A$,*

$$\rho(F(a)) \leq \rho(a) \quad \text{and} \quad \rho(F'(0)a) \leq \rho(a),$$

and

$$\sigma(F(a)) \cap \Gamma_{\rho(a)} = \sigma(F'(0)a) \cap \Gamma_{\rho(a)}.$$

Thus, under the hypotheses of Theorem 1, we may conclude that $G(a)$ always has the same peripheral spectrum as a , even if $a \in \Omega_A \setminus Z_A$. One is led naturally to ask if the same might be true of the whole spectrum.

Question 1. *Let A be a semisimple Banach algebra and let $G: \Omega_A \rightarrow \Omega_A$ be a holomorphic map satisfying $G(0) = 0$ and $G'(0) = I$. Does it follow that $\sigma(G(a)) = \sigma(a)$ for all $a \in \Omega_A$?*

If A is the algebra of $n \times n$ matrices, then the answer is yes: this was proved in [5, Theorem 3]. Even if the answer is no in general, perhaps it is still at least true that $\sigma(G(a))$ and $\sigma(a)$ have the same polynomial hull.

Combining the Schwarz lemma with the earlier Cartan theorem leads to our final result. We write e for the identity in a Banach algebra.

Theorem 4. *Let A and B be commutative semisimple Banach algebras, and let $F: \Omega_A \rightarrow \Omega_B$ be a biholomorphic map.*

- (a) *If $F(0) = 0$, then F extends to a linear isomorphism $F: A \rightarrow B$.*
 (b) *If also $F(e) = e$, then $F: A \rightarrow B$ is an algebra isomorphism.*

Part (b) of this result is originally due to M. Nagasawa [4] (see also [1, §4.1], which explains the connection with the Banach-Stone theorem). The proofs in both these references depend on convexity theory, whereas the complex-variable proof of Theorem 4 given below is quite different.

Once again, one is led to ask if this result has a non-commutative version. The most natural formulation seems to be the following, which has been an open question for about 25 years.

Question 2. *Let A and B be semisimple Banach algebras, and let $F: A \rightarrow B$ be a linear isomorphism such that $F(e) = e$ and $\sigma(F(a)) = \sigma(a)$ for all $a \in A$. Does it follow that F is a Jordan isomorphism, i.e. $F(a^2) = F(a)^2$ for all $a \in A$?*

The answer is known to be yes in a number of special cases (see [2], which also contains a brief history of the problem). It is possible that Question 2 might prove more tractable if we had a positive answer to Question 1.

2. PROOFS

Proof of Theorem 1. (a) Fix $c \in \Omega_A \cap Z_A$, and consider the function $\lambda \mapsto G(\lambda c)$, which is holomorphic on $\{\lambda \in \mathbb{C}: |\lambda| < 1/\rho(c)\}$. Since $G(0) = 0$ and $G'(0) = I$, this function has a Taylor expansion about 0 of the form

$$(1) \quad G(\lambda c) = \lambda c + \sum_{j=2}^{\infty} \lambda^j a_j \quad (|\lambda| < 1/\rho(c)),$$

where the a_j are elements of A . Our aim is to show that $a_j = 0$ for all j . Suppose, for a contradiction, that this is not the case, and let k be the smallest integer such that $a_k \neq 0$. Take $q \in A$ with $\rho(q) = 0$, and let $n \geq 1$. Then, writing G^n for the n -fold composition $G \circ \dots \circ G$, we have

$$(2) \quad G^n(\lambda c + \lambda^k nq) = \lambda c + \lambda^k n(a_k + q) + O(\lambda^{k+1}) \quad \text{as } \lambda \rightarrow 0.$$

Now as c and q commute, it follows that $\rho(\lambda c + \lambda^k nq) \leq \rho(\lambda c) + \rho(\lambda^k nq) = |\lambda|\rho(c)$ (see e.g. [1, Corollary 3.2.10]), and so we can define a holomorphic function $g: \{0 < |\lambda| < 1/\rho(c)\} \rightarrow A$ by

$$g(\lambda) = \frac{G^n(\lambda c + \lambda^k nq) - \lambda c}{n\lambda^k} \quad (0 < |\lambda| < 1/\rho(c)).$$

From (2) we see that the isolated singularity at $\lambda = 0$ can be removed by setting $g(0) = a_k + q$. By Vesentini's theorem (see e.g. [1, Theorem 3.4.7]), the composition $\rho \circ g$ is a subharmonic function on $\{|\lambda| < 1/\rho(c)\}$, and so by the maximum principle

$$\rho(g(0)) \leq \max_{|\lambda|=1} \rho(g(\lambda)).$$

Making use of [1, Corollary 3.2.10] again to estimate the right-hand side, it follows that

$$\rho(a_k + q) \leq 2/n.$$

As this is true for each n , we can let $n \rightarrow \infty$ deduce that $\rho(a_k + q) = 0$. And as this holds for each $q \in A$ with $\rho(q) = 0$, Zemánek's characterization of the radical [1, Theorem 5.3.1] implies that a_k belongs to the radical of A , which is zero since A is semisimple. Thus $a_k = 0$, and we have arrived at a contradiction. We conclude that indeed $a_j = 0$ for all $j \geq 2$, and hence from (1) that $G(c) = c$.

(b) Let $a \in \Omega_A \setminus Z_A$. Then there exists $u \in A$ such that $au \neq ua$. We can suppose that $\|u\| < 1$. Then $v := \log(e - u)$ satisfies $e^{-v}ae^v \neq a$. Choose a continuous linear functional ϕ on A such that $\phi(a) = 1$, and define $G: \Omega_A \rightarrow \Omega_A$ by

$$G(x) = e^{-\phi(x)v} x e^{\phi(x)v} \quad (x \in \Omega_A).$$

Then G is holomorphic, $G(0) = 0$ and $G'(0) = I$, but $G(a) = e^{-v}ae^v \neq a$. □

Proof of Theorem 2. Fix $c \in \Omega_A \cap Z_A$, and suppose also, for the moment, that $c \neq 0$. Then $F(c) \neq F(0)$, so there exists a continuous linear functional ψ on B such that $\psi(F(c) - F(0)) = 1$. Take $b \in B$, and define $C: \Omega_B \rightarrow \Omega_B$ by

$$C(y) = e^{-\psi(y-F(0))^2 b} y e^{\psi(y-F(0))^2 b} \quad (y \in \Omega_B).$$

Then C is holomorphic, $C(F(0)) = F(0)$ and $C'(F(0)) = I$. Hence if we set $G = F^{-1} \circ C \circ F$, then G is a holomorphic self-map of Ω_A satisfying $G(0) = 0$ and $G'(0) = I$. By Theorem 1, it follows that $G(c) = c$, so that $C(F(c)) = F(c)$ or, in other words $e^{-b} F(c) e^b = F(c)$. This holds for every $b \in B$, so as in the proof of part (b) of Theorem 1, we deduce that $F(c) \in Z_B$. This has been proved under the assumption that $c \neq 0$, but by continuity it remains true if $c = 0$. Thus $F(\Omega_A \cap Z_A) \subset \Omega_B \cap Z_B$, and the reverse inclusion follows by applying the same argument to F^{-1} . \square

Proof of Theorem 3. Fix $a \in \Omega_A$, and define $f: \{0 < |\lambda| < 1/\rho(a)\} \rightarrow B$ by

$$f(\lambda) = F(\lambda a)/\lambda \quad (0 < |\lambda| < 1/\rho(a)).$$

Then f is holomorphic, and since $F(0) = 0$ we can remove the isolated singularity at $\lambda = 0$ by setting $f(0) = F'(0)a$. Now using Vesentini's theorem once again, the function $\rho \circ f$ is subharmonic on $\{|\lambda| < 1/\rho(a)\}$, and so for each $r < 1/\rho(a)$ it follows from the maximum principle that

$$\rho(f(\lambda)) \leq \max_{|\mu|=r} \rho(f(\mu)) \leq 1/r \quad (|\lambda| \leq r).$$

Letting $r \rightarrow 1/\rho(a)$, we obtain

$$\rho(f(\lambda)) \leq \rho(a) \quad (|\lambda| < 1/\rho(a)).$$

In particular, taking $\lambda = 1$ and $\lambda = 0$, we deduce that $\rho(F(a)) \leq \rho(a)$ and $\rho(F'(0)a) \leq \rho(a)$, which proves the two inequalities in the theorem. (This half of the proof was essentially the same as in [5, Theorem 2].)

Now fix $\zeta \in \mathbb{C}$ with $|\zeta| = \rho(a)$. Then we have $\rho(f(\lambda) + \zeta e) \leq 2\rho(a)$ for all λ , with equality if and only if $\zeta \in \sigma(f(\lambda))$. However, by Vesentini's theorem once more, the function $\lambda \mapsto \rho(f(\lambda) + \zeta e)$ is subharmonic, so if it attains a maximum, then it must be constant. Therefore if $\zeta \in \sigma(f(\lambda))$ for *one* value of λ , then the same holds for *all* λ . The conclusion is that the set $\sigma(f(\lambda)) \cap \Gamma_{\rho(a)}$ is independent of λ . In particular, taking $\lambda = 1$ and $\lambda = 0$ we obtain $\sigma(F(a)) \cap \Gamma_{\rho(a)} = \sigma(F'(0)a) \cap \Gamma_{\rho(a)}$, which proves the final statement of the theorem. \square

Proof of Theorem 4. (a) By the second inequality of Theorem 3, applied both to F and F^{-1} , we see that $F'(0)|_{\Omega_A}$ is a biholomorphic map of Ω_A onto Ω_B . Hence if we set $G = F^{-1} \circ F'(0)|_{\Omega_A}$, then G is a holomorphic self-map of Ω_A satisfying $G(0) = 0$ and $G'(0) = I$. From Theorem 1, together with the fact that A is commutative, it follows that $G(a) = a$ for all $a \in \Omega_A$, and so $F \equiv F'(0)$ on Ω_A . Thus $F'(0)$ is a linear isomorphism of A onto B which extends F .

(b) Assume now that F has been so extended, and that $F(e) = e$. For $t \in (-1, 1)$, define $M_t: \Omega_A \rightarrow \Omega_A$ and $N_t: \Omega_B \rightarrow \Omega_B$ by

$$\begin{aligned} M_t(x) &= (x - te)(e - tx)^{-1} & (x \in \Omega_A), \\ N_t(y) &= (y - te)(e - ty)^{-1} & (y \in \Omega_B). \end{aligned}$$

Then M_t is biholomorphic, with $M_t(0) = -te$ and $M'_t(0) = (1 - t^2)I$, and similarly for N_t . Also, since F is linear and maps e to e , we have $F(-te) = -te$. Hence if we

define $G = F^{-1} \circ N_t^{-1} \circ F \circ M_t$, then G is a holomorphic self-map of Ω_A satisfying $G(0) = 0$ and $G'(0) = I$. Applying Theorem 1 once more, we deduce that $G(a) = a$ for all $a \in \Omega_A$, or in other words that $F(M_t(a)) = N_t(F(a))$ for all $a \in \Omega_A$. Now fix $a \in \Omega_A$, and expand both sides of this last equation in powers of t . This gives

$$F(a) + tF(a^2 - e) + O(t^2) = F(a) + t(F(a)^2 - e) + O(t^2) \quad \text{as } t \rightarrow 0,$$

and equating the coefficients of t we obtain $F(a^2) = F(a)^2$. Applying this with $a = \frac{1}{2}(a_1 \pm a_2)$, we deduce that $F(a_1 a_2) = F(a_1)F(a_2)$ for all $a_1, a_2 \in \Omega_A$, and hence, by homogeneity, for all $a_1, a_2 \in A$. This completes the proof. \square

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